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On ACTL Formulas Having Linear

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In case an ACTL formula ϕ fails over a transition graph M, it is most useful to provide a counterexample, i.e., a computation tree of M, witnessing the failure. If there exists a *single path* in M which by itself witnesses the failure of ϕ, then ϕ has a *linear counterexample*. We show that, given M and ϕ, where $M \not\models \phi$, it is NP-hard to determine whether there exists a linear counterexample. Moreover, it is PSPACE-hard to decide whether an ACTL formula ϕ always admits a linear counterexample if it fails. This means that there exists no simple characterization of the ACTL formulas that guarantee linear counterexamples. Consequently, we study *templates* of ACTL formulas, i.e., skeletons of modal formulas whose atoms are disregarded. We identify the (unique) maximal set **LIN** of templates whose instances (obtained by replacing atoms with arbitrary pure state formulas) always guarantee linear counterexamples. We show that for each ACTL formula ϕ which is an instance of a template $\gamma^\star \in \mathbf{LIN}$, and for each Kripke structure M such that $M \not\models \phi$, a single path of M witnessing the failure by itself can be computed in polynomial time.

Key Words: model checking; verification; counterexamples; linear counterexamples; counterpaths; temporal reasoning; ACTL; branching time logics.

1. INTRODUCTION

ACTL is a well-known particular fragment of Computational Tree Logic (CTL), which is a propositional branching-time temporal logic [2]; see [7, 6] for a rich

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background on this and further such logics. ACTL formulas are specified and evaluated over *Kripke structures* which model finite-state systems. Besides Boolean connectives, ACTL provides linear-time and branching time operators. The linear-time operators allow for expressing properties of a particular evolution of the systems given by a series of events in time. Branching time operators allow to take into account the existence of multiple possible future scenarios, starting from a given system state at a point in time. The temporal order defines an evolution tree, which branches from that point towards the future. Thus, every point in time has a unique past, but, in general, more than one future. Each branch of the tree amounts to a particular evolution series.

The elementary linear-time operators are **X** (*next time*), **U** (*until*), and **V** (*unless*, *releases*). Informally, $\mathbf{X}\phi$ means that ϕ is true at the next point in time; $\phi_1\mathbf{U}\phi_2$ means that there exists a prefix of the computation path such that ϕ_2 is true at the last state and ϕ_1 is true at all previous states of this prefix; and $\phi_1\mathbf{V}\phi_2$ means that truth of ϕ_1 releases truth of ϕ_2. Further operators such as $\mathbf{F}\phi$ (*sometimes* ϕ), $\mathbf{G}\phi$ (*always* ϕ) can be derived from the elementary operators. ACTL has the branching time operator **A**, by which it is possible to express *necessary* properties for an evolution tree. Informally, $\mathbf{A}\phi$ means that ϕ is true for all branches of the tree. Note that in full *CTL*, a dual operator **E** for expressing *possible* properties (true along some branch) is provided.

1.1. Counterpaths and Linear Counterexamples

The task of an automatic ACTL model checker is the verification of a given ACTL formula ϕ on a Kripke Structure M. In case M does not satisfy ϕ (denoted $M \not\models \phi$), advanced model checkers (e.g. McMillan's SMV system [11], or the debugger described by Hojati *et al.* [9]) provide more information. In particular, as a witness for the failure, a finite representation of an infinite computation path π of M is provided. This path represents a counterexample to ϕ in M. In the ideal case, such a path π witnesses *by itself* that $M \not\models \phi$, in other terms, all information needed to disprove that $M \models \phi$ is already contained in π. In this case, we call π a *counterpath.*

To make the above concepts precise, we give in Section 3 a formal definition of the concept of *counterexample.* Roughly, a counterexample to an ACTL formula ϕ on structure M is a computation tree represented as a *multi-path* disproving that $M \models \phi$. In case this multi-path has no true branching, and thus actually represents a unique path, we speak about a *linear counterexample.* A counterpath for ϕ in M is then the unique path corresponding to a linear counterexample. Note that if there exists such a counterpath π, then it holds that $M_\pi \not\models \phi$, where M_π is the Kripke structure induced by π, i.e., the structure whose states are all those states of M that also occur in π, where the states are, moreover, labeled by the same labels as in M, and whose transitions are those that occur in π.

EXAMPLE 1.1. Let M amount to the transition graph in Fig. 1, where initial states are colored black, and consider the formula $\phi = \mathbf{A}(true\mathbf{U}a_1)$, which can be abbreviated as $\mathbf{AF}a_1$.

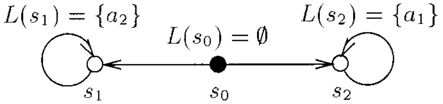

FIG. 1. Transition graph representing structure M (initial state s_0).

It holds that $M \not\models \phi$: Along the path $\pi = [s_0, s_1, s_1, ...]$, the atom a_1 is false at each stage $\pi(i)$ of π, $i \geqslant 0$. This implies $M, \pi \models \neg \mathbf{F} a_1$. Thus, π witnesses the failure of ϕ in M. Note that the information contained in π alone is sufficient for disproving ϕ; we do not have to consider elements of M (states or transitions) outside π to show that $M \not\models \phi$. Thus π is a counterpath of ϕ.

1.2. Linear Counterexamples May Not Exist

A counterpath provides very useful, compactly presented, and self-contained information to a system designer or verifier, allowing him or her to locate a design error in a most comfortable way. It would thus be most desirable to be able to compute a (representation of a) counterpath in polynomial time whenever an ACTL formula ϕ fails over a structure M.

Unfortunately, as shown by the example below, if $M \not\models \phi$, a counterpath (or, equivalently, a linear counterexample) does not necessarily exist.

Example 1.2. Let M amount to the transition graph in Fig. 2, and consider $\phi = \mathbf{A}(\mathit{true}\mathbf{U}\mathbf{A}(\mathit{false}\mathbf{V}a))$, which can be abbreviated as $\mathbf{AFAG}a$. It is easy to verify that $M \not\models \phi$. Indeed, there is a path $\pi = [s_0, s_0, ...]$ starting from the initial state where always the nested formula $\mathbf{AG}a$ does not hold, as, for each $i \geqslant 0$, there exists a path starting at $\pi(i)$ where sometimes a is not true (e.g., on the path $\pi' = [s_0, s_1, s_2, s_2, ...]$ a is not true at s_1). The path π itself is not a complete counterexample. To disprove that $M \models \phi$, it is necessary to consider a further path for each state of π (here always s_0) in order to show that the subformula $\mathbf{AG}a$ does not hold. This gives rise to a multi-path Π, which we write as follows: $\Pi = [[s_0, s_1, s_2, s_2, ...], [s_0, s_1, s_2, s_2, ...], ...]$. It consists of a computation tree with main branch $[s_0, s_0, ...]$ in which at each stage a branch $[s_0, s_1, s_2, s_2, ...]$ starts. This multi-path Π is a counterexample for ϕ in M, and not the single path π. Note that Π is *not* a linear counterexample, but a truly branching infinite tree. Note, furthermore, that no single path is a counterexample for ϕ. Therefore, no linear counterexample exists in this case, and thus no counterpath witnessing that $M \not\models \phi$ exists.

Besides the above very simple example, many other cases can be found in which each counterexample is a truly branching computation tree. They include formulas of the shape $\mathbf{AF}\phi \vee \mathbf{AF}\psi$ (e.g., $\mathbf{AF}a_1 \vee \mathbf{AF}a_2$ on the structure M in Fig. 1), $\mathbf{AF}(\mathbf{AG}\phi \vee \mathbf{AG}\neg\phi)$, which informally states that any evolution must commit at

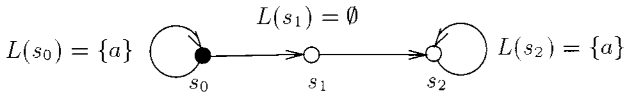

FIG. 2. Another transition graph representing structure M (initial state s_0).

some point about a condition ϕ being true or false, and $\mathbf{AF}\phi \vee \mathbf{AG}\psi$, which states that either ϕ becomes true at some stage or ψ always holds.

From these observations, we can infer that in many cases a simple "counter-example path" output by an ACTL model checker such as McMillan's system [11] can not be a *full* counterexample, but only *one path*—usually the main path or "backbone"—of a counterexample. Such a path may help to track the design or implementation error, but it does by itself not necessarily explain why the formula fails, and one may need to consider states and transitions outside that path in order to track the flaw. The debugger in [9] constructs a counterexample for an ACTL formula ϕ unwinding the formula. A counterpath would be desirable, since the unwinding can be done along it, without reference to other parts of the structure.

1.3. Main Research Questions Addressed

Given that linear counterexamples (and counterpaths) are useful, but do not always exist, the following questions naturally arise:

- Is there an efficient method of deciding whether an ACTL formula ϕ has a linear counterexample (and thus a counterpath) on a given Kripke structure M, where $M \not\models \phi$?
- Is there a simple characterization of those ACTL formulas which *guarantee* linear counterexamples? In other terms, is there an efficient method for telling whether a formula ϕ has the property that whenever $M \not\models \phi$ holds for a structure M, then there exists a linear counterexample (and thus a counterpath) witnessing this?
- If the above fails, how can we efficiently identify large classes of formulas that guarantee linear counterexamples?
- Can we efficiently *compute* linear counterexamples in case they exist (and, related to this, efficiently *recognize* them) ? If this is not generally possible, then maybe for large classes of ACTL formulas?

1.4. Main Results

Our main results are shortly summarized as follows:

- We give, in Section 2, a precise definition of the concepts of linear counterexample and of the related concept of *counterpath.*
- We show that given M and ϕ, where $M \not\models \phi$, it is NP-hard to determine whether there exists a linear counterexample (Theorem 4.2).
- As a consequence, even in case counterpaths exist, *computing* a counterpath is a hard problem. Therefore, unless $\mathrm{NP} = \mathrm{P}$, for every ACTL model-checker $\mathscr{MC}$ that works in polynomial time and produces "single-path counterexamples" in case of failure, there exist infinitely many Kripke structures M and formulas ϕ, such that $M \not\models \phi$ and the counterexample path output by $\mathscr{MC}$ represents a partial (and not a complete) counterexample even though there exists a counterpath (i.e., a path representing a complete counterexample).

TABLE 1

BNF Grammar for Linear Templates

$LIN ::= PSF \mid (LIN \wedge LIN) \mid (LIN \vee PSF) \mid (PSF \vee LIN) \mid \mathbf{AX}(LIN) \mid \mathbf{A}(PSF\,\mathbf{V}\,LIN) \mid ULIN$
$ULIN ::= \mathbf{A}(LIN\,\mathbf{U}\,PSF) \mid \mathbf{A}(PSF\,\mathbf{U}\,ULIN) \mid (ULIN \vee PSF) \mid (PSF \vee ULIN)$
$PSF ::= (PSF \wedge PSF) \mid (PSF \vee PSF) \mid \neg(PSF) \mid \bigstar$

• It is PSPACE-hard to decide whether an ACTL formula ϕ in case of failure always admits a linear counterexample (Theorem 4.1). This means that there exists no simple characterization of the ACTL formulas that guarantee linear counterexamples.

• Consequently, we study *templates* of ACTL formulas, i.e., skeletons of modal formulas whose atoms are disregarded and replaced by the symbol $\bigstar$. As main result of this paper, we identify the (unique) maximal set **LIN** of templates whose instances, obtained by replacing $\bigstar$'s with arbitrary pure state formulas, always guarantee linear counterexamples (Theorem 4.3). The set **LIN** of templates is given by the BNF grammar in Table 1. For example, the templates $\mathbf{AX}(\bigstar)$, $\mathbf{A}(\bigstar\mathbf{V}\mathbf{AX}(\bigstar)))$, and $(\bigstar \wedge \mathbf{A}(\bigstar\mathbf{V}\mathbf{AX}(\bigstar)))$ are in **LIN**, as well as $\mathbf{A}(\bigstar\mathbf{U}\bigstar)$, $\mathbf{A}(\bigstar\mathbf{U}\mathbf{A}(\bigstar\mathbf{U}\bigstar))$, and $\mathbf{A}(\mathbf{A}(\bigstar\mathbf{V}\mathbf{AX}(\bigstar))\mathbf{U}(\bigstar \wedge \bigstar))$. On the other hand, the template $\mathbf{A}(\bigstar\mathbf{U}\mathbf{A}(\bigstar\mathbf{V}\bigstar))$ of the formula $\phi = \mathbf{A}(true\mathbf{U}\mathbf{A}(false\mathbf{V}a))$ in Example 1.2 is not in **LIN**, and also the template $\mathbf{A}(\bigstar\mathbf{U}\bigstar) \vee \mathbf{A}(\bigstar\mathbf{U}\bigstar)$ of the formula $\mathbf{A}(true\mathbf{U}a_1) \vee \mathbf{A}(true\mathbf{U}a_2) = \mathbf{AF}a_1 \vee \mathbf{AF}a_2$ mentioned above is not in **LIN**.

Obviously, it is recognizable in polynomial time (and in fact in linear time) whether a template belongs to **LIN**, and whether an ACTL formula ϕ is an instance of some template in **LIN**. In particular, we prove:

∗ If ϕ is an instance of a template $\gamma^\bigstar \in \mathbf{LIN}$, then, for each structure M such that $M \not\models \phi$, there exists a linear counterexample, and thus a counterpath in M witnessing this failure.

∗ If $\gamma^\bigstar$ is a template not contained in **LIN**, then there exist an instance ϕ of $\gamma^\bigstar$ and a structure M such that $M \not\models \phi$ but there exists no linear counterexample for ϕ in M.

• We show that for each ACTL formula ϕ which is an instance of a template $\gamma^\bigstar \in \mathbf{LIN}$, and for each Kripke structure M such that $M \not\models \phi$, a counterpath, i.e., a single path of M witnessing the failure, can be computed in polynomial time (Theorem 5.2).

• Finally, we show that recognizing a valid counterpath for an arbitrary ACTL formula ϕ is possible in polynomial time. This follows from the fact that the problem can be easily reduced to a model checking problem $M' \models \phi$, which can be solved in polynomial time (Theorem 5.3).

Note that it could be the case that systems like McMillan's do always yield a valid counterpath in case the input formula ϕ is an instance of a template in **LIN**, i.e., they would be (sound and) *complete* for generating counterpaths on the class of **LIN** instances. Our results may serve as a starting point for determining the exact ACTL fragments on which such systems are complete with respect to generation of counterpaths. Furthermore, the a priori knowledge that linear counterexamples do always exist for instances of **LIN** templates may be exploited in the design of more efficient algorithms than those which handle the general case of arbitrary ACTL formulas like the one employed in McMillan's system. These issues are beyond the scope of the paper and left for further work.

1.5. Structure of the Paper

After this introduction, some preliminaries and notation are given in Section 2. In Section 3, the formal definition of counterexamples is provided, for which multi-paths are introduced. Thereafter, we turn our attention in Section 4 to linear counterexamples and multi-paths. After proving that recognizing linear ACTL formulas is intractable, we define the class **LIN** of templates; furthermore, we formally state the characterization of *c*-linear templates, which is the first main result of this paper. Sections 5–6 are devoted to the proof of this result and to the computation of counterpaths for **LIN**-instances, which is the second main result. The paper is closed in Section 7 with a discussion and an outlook on future work.

2. PRELIMINARIES

Definition 2.1 (ACTL formulas). Let A be a set of atomic propositions. Then, ACTL is the set of *state formulas on A* inductively defined as follows:

(1) Any Boolean formula over atoms from A built using the connectives $\wedge$, $\vee$, and $\neg$ is a *pure state formula*; a pure state formula is a *state formula*;

(2) if ϕ and ψ are state formulas, then $(\phi \vee \psi)$, and $(\phi \wedge \psi)$ are *state formulas*;

(3) if ϕ and ψ are state formulas, then $\mathbf{X}\phi$, $\phi\mathbf{U}\psi$ and $\phi\mathbf{V}\psi$ are *path formulas*;

(4) if ϕ is a path formula, then $\mathbf{A}(\phi)$ is a *state formula.*

Intuitively, path formulas describe properties of evolution series because they use temporal operators next time, until, and unless.

Notation. For any sets D_1 and D_2 of formulas, we shall use the following notation:

$$\begin{aligned}
\mathbf{AX}(D_1) &= \{\mathbf{AX}(\gamma) \mid \gamma \in D_1\},\\
\mathbf{AU}(D_1, D_2) &= \{\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2) \mid \gamma_1 \in D_1, \gamma_2 \in D_2\},\\
\mathbf{AV}(D_1, D_2) &= \{\mathbf{A}(\gamma_1 \mathbf{V} \gamma_2) \mid \gamma_1 \in D_1, \gamma_2 \in D_2\},\\
D_1 \wedge D_2 &= \{(\gamma_1 \wedge \gamma_2) \mid \gamma_1 \in D_1, \gamma_2 \in D_2\},\\
D_1 \vee D_2 &= \{(\gamma_1 \vee \gamma_2) \mid \gamma_1 \in D_1, \gamma_2 \in D_2\}.
\end{aligned}$$

Given a formula ϕ or a set S of formulas, we will denote by $AP(\phi)$ (resp., $AP(S)$) the set of atomic propositions occurring in ϕ (resp., S). We will use *true* and *false* as shorthand for pure state formulas which are tautologies and contradictions, respectively. We shall omit or add parentheses in formulas following the usual conventions.

The formal definition of the semantics of ACTL refers to particular *Kripke structures*. Informally, they are finite transition graphs with labeled states.

Definition 2.2 (Kripke Structure). A *Kripke structure* is a quintuple $M = (A, S_0, S, R, L)$ such that:

- A is a finite set of atomic propositions, denoted $A(M)$;
- S is a finite set of states, denoted $S(M)$;
- $S_0 \subseteq S$ is a finite set of initial states, denoted $S_0(M)$;
- $R \subseteq S \times S$ is a transition relation, denoted $R(M)$;
- $L: S \to 2^A$ is a mapping assigning to each state of S the set of atomic propositions true in that state; L is called *label function*, and is denoted by $L(M)$.

For convenience, we often denote by M_s the Kripke structure which is identical to M except $S_0(M_s) = \{s\}$ where $s \in S(M)$, i.e., s is the unique initial state. Furthermore, we will sometimes focus on structures M such that $S_0(M) = \{s_0\}$ and $(s, s_0) \notin R(M)$, for all $s \in S(M)$, i.e., M has a unique initial state s_0, and s_0 is not reachable from any state in M. We refer to such structures as *conic*.

Note that many authors (e.g. [7, 10]) require that the transition relation $R(M)$ in a Kripke structure M is *total*, i.e. $\forall s\, \exists s'. R(s, s')$ holds. This restriction would let the main results of this paper unaffected. We shall come back to this issue and discuss it in more detail in Section 7.

The dynamic temporal evolution is modeled by infinite paths in the Kripke structure.

Definition 2.3 (path). A *path* π of a Kripke structure M is an infinite sequence $\pi = [s_0, s_1, ..., s_i, ...]$ such that for each $i \geqslant 0$ $(s_i, s_{i+1}) \in R(M)$. Given an integer $i \geqslant 0$ and a path π we denote by $\pi(i)$ the $(i+1)$-th state of π.[1] Thus, the first state of a path π is denoted by $\pi(0)$. Given an integer $j \geqslant 0$ and a path π, the j-suffix π^j of π is the path $[\pi(j), \pi(j+1), ...]$. Clearly, $\pi = \pi^0$ and $\pi(i) = \pi^i(0)$.

The semantics of ACTL is now defined through an entailment relation $\models$, which can be applied on states s and paths π for evaluating state and path formulas, respectively.

Definition 2.4 (satisfaction). Let s and π be a state and a path in M, respectively. Then, the satisfaction relation $\models$ for state and path formulas, respectively, on a Kripke structure M is inductively defined as follows.

1. $M, s \models p$, if $p \in L(M)(s)$ for any atomic proposition $p \in A$;
2. $M, s \models \neg\phi$, if $M, s \not\models \phi$ where ϕ is a state formula;

[1] Thus, the first state of a path π is denoted by $\pi(0)$.

3. $M, s \models \phi_1 \vee \phi_2$, if $M, s \models \phi_1$ or $M, s \models \phi_2$ where ϕ_1 and ϕ_2 are state formulas;

4. $M, s \models \phi_1 \wedge \phi_2$, if $M, s \models \phi_1$ and $M, s \models \phi_2$ where ϕ_1, ϕ_2 are state formulas;

5. $M, s \models \mathbf{A}(\psi)$, if $M, \pi \models \psi$ for all paths π in M such that $\pi(0) = s$;

6. $M, \pi \models \mathbf{X}\phi$, if $M, \pi(1) \models \phi$;

7. $M, \pi \models \phi_1 \mathbf{U} \phi_2$, if there exists an integer $k \geqslant 0$ such that $M, \pi(k) \models \phi_2$ and $M, \pi(j) \models \phi_1$ for all $0 \leqslant j < k$;

8. $M, \pi \models \phi_1 \mathbf{V} \phi_2$, if for every $k \geqslant 0$ it holds that $M, \pi(j) \not\models \phi_1$ for all $0 \leqslant j < k$ implies $M, \pi(k) \models \phi_2$.

We write $M \models \phi$ if $M, s_0 \models \phi$, for every initial state $s_0 \in S_0(M)$.

Intuitively, a state formula holds along a path, if it is true at its first state; $\phi_1 \mathbf{U} \phi_2$ is true, if ϕ_1 is true along the path until some state is reached at which ϕ_2 is true; and $\phi_1 \mathbf{V} \phi_2$ is true, if there is no stage such that ϕ_2 is false and ϕ_1 is false at all previous states. Note that **U** and **V** are dual operators: $\phi_1 \mathbf{U} \phi_2$ is true precisely if $\neg\phi_1 \mathbf{V} \neg\phi_2$ is false.

3. MULTI-PATHS AND COUNTEREXAMPLES

If an ACTL formula ϕ is not true in a structure M, then there must be some evidence which proves the failure of the formula. For a pure state formula ϕ, an initial state s_0 at which ϕ is false is a witness of this fact; if ϕ is of the form $\mathbf{AX}\psi$, where ψ is a pure state formula,then a path π starting at some $s_0 \in S_0$ such that ψ is false at $\pi(1)$ is such a witness. The falsity of formulas $\mathbf{A}(\phi_1 \mathbf{U} \phi_2)$, $\mathbf{A}(\phi_1 \mathbf{V} \phi_2)$ where the ϕ_i are pure state formulas is witnessed similarly by a path π.

Intuitively, a path π as described is a *counterexample* for the truth of ϕ in M. It appears that for more complex formulas ϕ which involve nested **A** quantifiers, a single path π may not be by itself witness that ϕ fails in M. To formally capture this, nesting of paths must be taken into account. This motivates the definition of multi-paths, which serve as a basis for a formal definition of counterexamples [1].

3.1. Multi-Paths

Informally, a multi-path represents an infinite tree T, which has a designated branch as a backbone (called *main path*). The branches of the tree which spring off from the main path at a certain stage are collected in a tree, which is recursively represented as a multi-path. Thus, multi-paths can be inductively defined. Observe that this representation of a tree is different from the usual inductive definition in which a tree is built by assigning child nodes to a parent node. The main advantage of the multi-path concept is the preservation of the nesting of paths, which is lost in the standard tree definition.

Preliminary to the formal definition of multi-paths, we introduce multi-sequences.

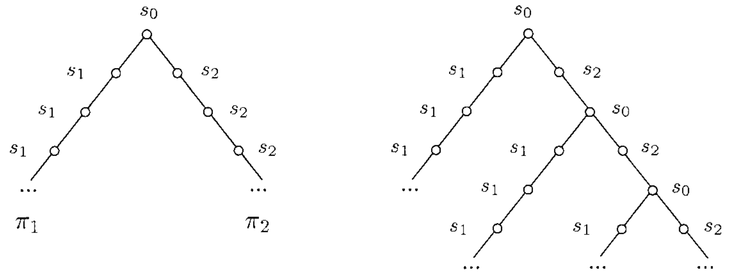

FIG. 3. Branching paths.

DEFINITION 3.1 (multi-sequence). Let S be a set of states. Then,

- for every state $s \in S$, $\Pi = s$ is a finite multi-sequence in S;
- if $\Pi_0, \Pi_1, \ldots$ are countably infinitely many multi-sequences in S, then $\Pi = [\Pi_0, \Pi_1, \ldots]$ is a multi-sequence in S.

For any multi-sequence Π, its $(i+1)$-th element is denoted by $\Pi(i)$, for all $i \geqslant 0$;[2] moreover, its *origin*, denoted $or(\Pi)$, is $or(\Pi) = s$, if $\Pi = s$ is a single state, and $or(\Pi) = or(\Pi(0))$, otherwise.

Next we introduce the notion of *main sequence* of a multi-sequence. Informally, it is the sequence formed by the origins of all elements in a multi-sequence.

DEFINITION 3.2 (main-sequence). For any multi-sequence Π, the *main sequence of* Π, denoted by $\mu(\Pi)$, is

- s, if $\Pi = s$ is finite;
- the sequence $[or(\Pi(0)), or(\Pi(1)), or(\Pi(2)), \ldots]$, otherwise.

Multi-paths are multi-sequences which model nested paths in M.

DEFINITION 3.3 (multi-path). A multi-sequence Π is a *multi-path* in M, if either Π is finite, or $\mu(\Pi)$ is a path in M and for every $i \geqslant 0$, $\Pi(i)$ is a multi-path in M. The main sequence of a multi-path Π is called the *main path* of Π.

Note that multi-paths generalize paths. Indeed, a path can be seen as an infinite multi-path Π such that each element $\Pi(i)$ is a state.

An infinite multi-path Π represents intuitively an evolving computing tree, whose branches are the main path $\mu(\Pi)$ and all paths of form $\pi_0\pi_1$ where $\pi_0 = \mu(\Pi)(0), \ldots, \mu(\Pi)(i-1)$ is a finite prefix of $\mu(\Pi)$ and π_1 is a branch of the multi-path $\Pi(i)$, where $\Pi(i)$ must be infinite.

EXAMPLE 3.1. Assuming proper M, the multi-sequence $\Pi = [[s_0, s_1, s_1, \ldots], s_2, s_2, \ldots]$ is a multi-path, which represents two paths $\pi_1 = [s_0, s_1, s_1, \ldots]$ and $\pi_2 = [s_0, s_2, s_2, \ldots]$ starting at s_0 (Fig. 3). π_2 is the main path $\mu(\Pi)$ of Π. The multi-path $\Pi = [[s_0, s_1, s_1, \ldots], s_2, [s_0, s_1, s_1, \ldots], s_2, [s_0, s_1, s_1, \ldots], \ldots]$ has main path $\mu(\Pi) = [s_0, s_2, s_0, s_2, \ldots]$ and represents the computation tree in which from $\mu(\Pi)$ at every even stage $\mu(\Pi)(2k)$ a path $[s_0, s_1, s_1, \ldots]$ branches off; hence, Π contains besides $\mu(\Pi)$ all paths of form $[(s_0, s_2)^i, s_0, s_1, s_1, \ldots]$, $i \geqslant 0$.

[2] Thus, $\Pi(0)$ is the first element of the multi-sequence Π.

An important note is that in general, a multi-path Π may not directly reflect in its structure a truly branching computation tree. In fact, the definition allows fake branching, in the sense that two nested branching paths may amount to the same path in the structure. For example, in the multi-path $\Pi = [s_0, s_1, [s_2, s_3, s_4, ...], s_3, s_4, ...]$, the branch $s_2, s_3, s_4, ...$ is identical to the remainder of the main path $s_2, s_3, s_4, ...$. This is not a shortcoming of our definition, but an important feature; it allows to express that a particular path is a subpath of another one. In an extended vocabulary for multi-paths, this could be expressed more elegantly; however, we disregard such an extension here. Note that for our purposes, we can restrict to multi-paths which have effective finite representations [1].

3.2. *Counterexamples*

We are now prepared to formalize the notion of counterexample. Intuitively, a counterexample for a formula ϕ is a special multi-path Π originating at an initial state demonstrating the falsity of ϕ. Since counterexamples are defined inductively, we need the concept of a local counterexample, which may origin at an arbitrary state rather than an initial state. For the technical definition of local counterexamples, we use an operation for merging two multi-paths into a single one.

Definition 3.4 (merge). Let Π_1 and Π_2 be two multi-paths such that $or(\Pi_1) = or(\Pi_2)$. The *merge* of Π_1 and Π_2, denoted by $\Pi_1 * \Pi_2$, is the multi-path recursively defined as follows:

$$\Pi_1 * \Pi_2 = \begin{cases} \Pi_1 & \text{if} \quad \Pi_2 \text{ is finite;} \\ [\Pi_1 * \Pi_2(0), \Pi_2(1), \Pi_2(2), ...] & \text{otherwise.} \end{cases}$$

Intuitively, the trees represented by Π_1 and Π_2 are merged at their common root.

Example 3.2. Merging $\Pi = [[s_0, s_{1_1}, s_{1_2}, ...], s_{2_1}, s_{2_3}, ...]$ and $\Pi' = [s_0, s_{3_1}, s_{3_2}, ...]$ yields

$$\Pi * \Pi' = [\Pi, s_{3_1}, s_{3_2}, ...] = [[[s_0, s_{1_1}, s_{1_2}, ...], s_{2_1}, s_{2_2}, ...], s_{3_1}, s_{3_2}, ...], \quad \text{while}$$

$$\begin{aligned} \Pi' * \Pi &= [\Pi' * [s_0, s_{1_1}, s_{1_2}, ...], s_{2_1}, s_{2_2}, ...] \\ &= [[\Pi', s_{1_1}, s_{1_2}, ...], s_{2_1}, s_{2_2}, ...] \\ &= [[[s_0, s_{3_1}, s_{3_2}, ...], s_{1_1}, s_{1_2}, ...], s_{2_1}, s_{2_2}, ...]. \end{aligned}$$

The two merges essentially represent the same branching of three paths $\pi_i = [s_0, s_{i_1}, s_{i_2}, ...]$ for $i \in \{1, 2, 3\}$, starting from s_0.

Note that $\mu(\Pi_1 * \Pi_2) = \mu(\Pi_2)$ in case Π_2 is infinite and $\mu(\Pi_1 * \Pi_2) = \mu(\Pi_1)$ otherwise. We remark that merging Π_1 and Π_2 by adding Π_1 as first element to Π_2 does not work, since in general, this leads to a set of paths different from those in Π_1 and Π_2; the result may even not be a multi-path.

Definition 3.5 (ℓ-counterexample). Let M be a Kripke structure and ϕ be an ACTL formula on $A(M)$. A multi-path Π in M is a *local* (ℓ-)*counterexample for* ϕ if, depending on the structure of ϕ, the following holds:

- if ϕ is a pure state formula: $\Pi = s$ is a state and $M, s \not\models \phi$;
- otherwise, if

1. $\phi = \mathbf{A}(\phi_1 \mathbf{U} \phi_2)$: Π is an infinite multi-path and either

1.1 there exists $k \geqslant 0$ such that $\Pi(k)$ is an ℓ-counterexample for $\phi_1 \vee \phi_2$, $\Pi(i)$ is an ℓ-counterexample for ϕ_2, for each $0 \leqslant i < k$, and $\Pi(j)$ is a state, for $j > k$; or

1.2 $\Pi(i)$ is a ℓ-counterexample for ϕ_2, for each $i \geqslant 0$;

2. $\phi = \mathbf{A}(\phi_1 \mathbf{V} \phi_2)$: Π is an infinite multi-path and there exists a k such that every $\Pi(j)$, $0 \leqslant j < k$, is an ℓ-counterexample for ϕ_1, $\Pi(k)$ is an ℓ-counterexample for ϕ_2, and every $\Pi(m)$ is a state, for $m > k$;

3. $\phi = \mathbf{AX}\phi_1$: Π is an infinite multi-path, $\Pi(1)$ is an ℓ-counterexample for ϕ_1, and $\Pi(i)$ is a state, for each $i \neq 1$;

4. $\phi = \phi_1 \vee \phi_2$: $\Pi = \Pi_1 * \Pi_2$, where Π_i, $i = 1, 2$, is an ℓ-counterexample for ϕ_i;

5. $\phi = \phi_1 \wedge \phi_2$: Π is an ℓ-counterexample for either ϕ_1 or ϕ_2.

Recall that $M \not\models \phi$ if there exists an initial state s_0 at which ϕ is false. Hence, we introduce a notion of "global" counterexample.

Definition 3.6 (counterexample). Let M be a Kripke structure and ϕ be a formula on $A(M)$. Any ℓ-counterexample Π for ϕ in M such that $or(\Pi) \in S_0(M)$ is called a *counterexample for* ϕ *in* M.

Example 1.1 illustrates this definition. Let us consider some more examples.

Example 3.3. Reconsider the Kripke structure M from Fig. 1, and let $\psi = \mathbf{A}(\mathit{false}\mathbf{V}\mathbf{A}(\mathit{true}\mathrm{U}a_1))$. Also this formula is false on M. Intuitively, this is witnessed by the path $\pi = [s_0, s_1, s_1, ...]$ again. However, from the formal definition, π is not a counterexample of ψ, as it does not respect witness paths for the subformula $\mathbf{A}(\mathit{true}\mathrm{U}a_1)$ of ψ. The multi-path $\Pi = [[s_0, s_1, ...], s_1, s_1, ...]$ is a proper counterexample for ψ according to the definition, as well as any multi-path $[s_0, (s_1,)^i, [s_1, s_1, ...], s_1, s_1, ...]$, where $i \geqslant 0$.

Finally, also the formula $\rho = \mathbf{A}(\mathit{true}\mathbf{U}\mathbf{A}(\mathit{false}\mathbf{V}a_1))$ is false in M; again, intuitively the path $\pi = [s_0, s_1, s_1, ...]$ shows this. Formally, the multi-path $[[s_0, s_1, s_1, ...], [s_1, s_1, ...], [s_1, s_1, ...], ...]$ is a counterexample for ρ; in fact, it is the unique counterexample.

The following result states that ℓ-counterexamples appropriately model the failure of a formula in a state.

THEOREM 3.1 [1]. Let M be a Kripke structure, ϕ a formula on $A(M)$, and $s \in S(M)$. Then, $M, s \not\models \phi$ if and only if there exists an ℓ-counterexample Π for ϕ such that $or(\Pi) = s$.

COROLLARY 3.2 [1]. *For any Kripke structure M and formula ϕ on $A(M)$, $M \not\models \phi$ if and only if there exists a counterexample Π for ϕ in M.*

As discussed earlier, in many cases a counterexample for a formula is (essentially) a single path. This is true e.g. for the formulas considered in the Examples 1.1 and 3.3. However, as Example 1.2 and the following example show, there are different cases in which a truly branching tree is needed.

EXAMPLE 3.4. Consider the structure M as in Fig. 1 again, but now the formula $\phi = \mathbf{A}(true\mathbf{U}a_1) \vee \mathbf{A}(true\mathbf{U}a_2)$. Clearly, $M \not\models \phi$: For every a_i, $i = 1, 2$, there is an infinite path $\pi_i = s_0, s_i, s_i, ...$ which never reaches a state at which a_i is true; hence, every disjunct $\mathbf{AF}a_i$ in ϕ is false. A counterexample for ϕ is the multi-path $\Pi = [[s_0, s_1, s_1, ...], s_2, s_2, ...]$, which results by merging the π_i's into $\Pi = (\pi_1 * \pi_2)$. Notice that no counterexample for ϕ exists that is an ordinary path, and that $\pi_1 * \pi_2$, $\pi_2 * \pi_1$ are the only (isomorphic) counterexamples for ϕ.

4. LINEAR COUNTEREXAMPLES

In this section, we formalize our intuition of a single path counterexample from the previous section. For this purpose, we introduce first the concept of a linear multi-path. Such a path is built over a single path in the structure, which exactly prescribes the next state in each transition throughout the multi-path.

4.1. *Linear Counterexamples and c-Linear Formulas*

DEFINITION 4.1 (linear multi-path). A multi-path Π is *linear*, if one of the following applies:

1. Π is finite (i.e., a single state); or
2. for each $i \geqslant 0$, either
   - 2.1 $\Pi(i)$ is a state, or
   - 2.2 $\mu(\Pi(i))$ coincides with $\mu(\Pi)^i$ (the i-suffix of $\mu(\Pi)$) and $\Pi(i)$ is linear.

Informally, a multi-path is linear if the main paths of its elements are suffixes of its main path, and this is recursively true also for the multi-paths of the sequence. Thus, while in general, multi-paths represent evolutions with branching, linear multi-paths have only artificial branching, and represent essentially a single path.

EXAMPLE 4.1. Consider the multi-path

$$\Pi = [s_0, s_1, s_2, s_3, [s_4, s_5, s_4, [s_5, s_4, s_5, s_4, ...], s_4, s_5, ...], s_5, s_4, s_5, s_4, ...].$$

As can be seen, this multi-path is linear. The path $[s_5, s_4, s_5, s_4, ...]$ nested into $\Pi(4)(3)$ represents a path branching from the main path of $\Pi(4)$. However, this path coincides with the suffix $\mu(\Pi(4))^3$ of the main path of $\Pi(4)$. Hence, it does not represent an alternative evolution. In this sense, a linear multi-path represents only linear evolutions.

Observe that the multi-path $\Pi' = [[s_0, s_1, s_2, s_3, s_2, s_3, ...], s_4, s_5, s_6, s_5, s_6, s_5, ...]$ is not linear.

We remark that we could have equivalently defined linear multi-paths in terms of bisimilarity of branching computations. Recall that two processes are (weakly) *bisimilar* [12], if there exists some *bisimulation* on them, i.e., a binary relation $\mathscr{B}$ on processes such that whenever $\mathscr{B}(P, Q)$ and P can perform some transition α to become P', then Q can perform the same transition α to become some Q' such that $\mathscr{B}(P', Q')$ holds, and, vice versa, if Q can become Q' by some transition α, then P can become some P' by transition α such that $\mathscr{B}(P', Q')$ holds. Every infinite multi-path Π (thus, also every path) represents a process P that can become the process $[\Pi(1), \Pi(2), ...]$ by the transition $\alpha = or(\Pi)$ and any process P' that $\Pi(0)$ can become (by the same transition). We may then call an infinite multi-path Π linear, if it is bisimilar to some (simple) path π. This notion of linearity is, as easily seen, equivalent to the one in Definition 4.1; in fact, under this notion Π is linear if and only if it is bisimilar to its main path $\mu(\Pi)$.

Definition 4.2 (linear counterexample and counterpath). A counterexample Π for an ACTL formula ϕ in a structure M is *linear*, if Π is a linear multi-path. The main path $\mu(\Pi)$ of any linear counterexample Π for ϕ in M is a *counterpath* for ϕ in M.

As easily verified, the counterexamples for the formulas presented in Examples 1.1 and 3.3 are linear counterexamples, and the "intuitive" counterexamples there are the respective counterpaths.

As for counterexamples, it is of particular interest to have a linear counterexample at hand, since such a counterexample is in generally easier to understand than an arbitrary counterexample. Moreover, the description of such counterexamples can be simplified. Observe that McMillan's SMV procedure [11] returns a single path π rather than a counterexample as used here when an ACTL formula fails. This path plays a similar role as the main path of our notion of a counterexample Π. If Π and π grasp the same witness, then $\mu(\Pi)$ should coincide with π, and it contains in fact all relevant information which is needed for witnessing the failure of ϕ. From π, a counterexample respecting the (artificial) branching of paths as required from the structure of ϕ can be reconstructed.

We thus direct our attention to the existence of linear counterexamples.

Definition 4.3 (c-linear). An ACTL formula ϕ is *c-linear*, if $M \not\models \phi$ implies that a linear counterexample for ϕ exists in M, for every Kripke structure M.

4.2. Complexity of Recognizing c-Linear Formulas

Unfortunately, recognizing c-linear formulas is complex in general, which is expressed by the following result.

THEOREM 4.1. *Deciding whether a given* ACTL *formula* ϕ *is c-linear is* PSPACE-*hard.*

Proof. This result is proved by a reduction from the unsatisfiability problem for ACTL formulas on structures M where $R(M)$ is total. This problem is PSPACE-complete by results of Kupferman and Vardi (see [10]).

Let ϕ be an arbitrary ACTL-formula, and let a be a fresh atom not occurring in ϕ. Let the formula ψ be defined as follows:

$$\psi = \mathbf{AX}a \vee \mathbf{AX}(\neg a \wedge \phi).$$

It holds that ψ is c-linear if and only if ϕ is unsatisfiable over structures M where $R(M)$ is total.

To prove this, suppose first that ϕ is unsatisfiable over all M where $R(M)$ is total. Let M be any structure (where $R(M)$ is not necessarily total) such that $M \not\models \psi$. This implies that $\mathbf{AX}a$ has a counterexample in M, which is a simple path π represented by a pair P, C where P is a path (prefix) and C a cycle in M. The assumption on ϕ implies that $\neg a \wedge \phi$ is globally false (and in particular, at $\pi(1)$) in the structure M_π which is naturally induced by π in M. Consequently, π is a counterpath for ψ in M_π, and thus also in M. This means that ψ is c-linear.

Now suppose that ϕ is satisfiable on some structure M' with total $R(M')$. Hence, a state $s'_0 \in S_0(M')$ exists such that $M', s_0' \models \phi$. Let M be the structure corresponding to the transition graph in Fig. 4.

It holds that $M \not\models \psi$. Indeed, every path $\pi_1 = [s_0, s'_0, ...]$ is a counterpath for $\psi_1 = \mathbf{AX}a$, and the path $\pi_2 = [s_0, s_1, s_1, ...]$ is a counterpath for $\psi_2 = \mathbf{AX}(\neg a \wedge \phi)$; thus, their merge $\Pi = \pi_1 * \pi_2$ is a counterexample for ψ. Clearly, any counterexample for ψ in M must contain both s'_0 and s_1; thus, a linear counterexample for ψ in M is impossible, which means that ψ is not c-linear. ∎

This result implies that a polynomial-size and polynomial-time checkable proof witnessing that a formula is c-linear is illusive, and thus we may abandon the search for an appealing syntactical characterization of c-linear formulas.

A related, in practice perhaps more important issue is whether the existence of a linear counterexample for a formula can be efficiently decided *ad hoc*, i.e., given an ACTL formula ϕ and a structure M, decide whether ϕ has a linear counterexample in M (and, if so, return a counterpath represented in a suitable way). As it turns out, also this problem is intractable.

THEOREM 4.2. *Given a Kripke structure M and an* ACTL-*formula ϕ, deciding whether ϕ has a linear counterexample* (*equivalently, a counterpath*) *in M is* NP-*hard.*

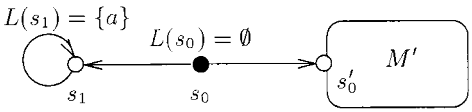

FIG. 4. Structure M for $\psi = \mathbf{AX}a \vee \mathbf{AX}(\neg a \wedge \phi)$ (initial state s_0).

Proof. We describe a polynomial-time transformation of deciding whether a given directed graph $G=(V, E)$ has a Hamiltonian circuit, which is well-known NP-complete [8], into this problem. Recall that a Hamiltonian circuit is a sequence $C=v_{i_1}, ..., v_{i_n}$ of all the vertices $V=\{v_1, ..., v_n\}$ such that an edge is directed from v_{i_j} to $v_{i_{j+1}}$ and from v_{i_n} to v_{i_1}.

We construct M and ϕ as follows. The set S of states of M is V, which is also the set A of atomic propositions and the set S_0 of initial states. The transition relation R is E, and each $v \in V$ has the label $L(v)=\{v\}$.

The formula ϕ is as follows:

$$\phi=\mathbf{A}\left(\mathit{true}\mathbf{U}\left(\bigvee_{v \in V}\left(v \wedge \bigvee_{w \in V\setminus\{v\}} \mathbf{AXA}(v\mathbf{V}\neg w)\right)\right)\right).$$

Intuitively, a linear counterexample for ϕ in M is an infinite path π such that in each state $\pi(i)=v$, the path must be continued in states $\pi(i+1)$, $\pi(i+2)$, ..., such that all other vertices $w \neq v$ appear before v may reappear.

We claim that G has a Hamiltonian circuit if and only if ϕ has a counterpath in M.

($\Rightarrow$) Let $C=v_{i_1}, ..., v_{i_n}$ be a Hamiltonian circuit of G. We claim that the path $\pi=(v_{i_1}, v_{i_2}, ..., v_{i_n})^{\infty}$ is a counterpath of ϕ. To verify this, we have to show that the formula

$$\bigvee_{v \in V} \psi_v, \qquad \text{where} \quad \psi_v=v \wedge\left(\bigvee_{w \in V\setminus\{v\}} \mathbf{AXA}(v\mathbf{V}\neg w)\right)$$

is false in each state $\pi(i)$, $i \geqslant 0$, and that a local counterexample witnessing this fact can be built over π^i.

For each $v \in V$ such that $v \neq \pi(i)$, v is false at $\pi(i)$ and thus $\pi(i)$ is a local counterexample for ψ_v over π^i. For the $v \in V$ such that $v=\pi(i)$, we must show that for each $w \in V\setminus\{v\}$, the suffix π^i is a local counterpath of the formula $\mathbf{AXA}(v\mathbf{V}\neg w)$; that is, that the suffix π^{i+1} is a local counterexample of $\mathbf{A}(v\mathbf{V}\neg w)$. Clearly, this is true for the $w \in V\setminus\{v\}$ such that $w=\pi(i+1)$; any $w' \in V\setminus\{v, w\}$ occurs as $\pi(i+k)$, where $1<k<n$, and v is false at $\pi(i+k-1)$; thus, π^{i+1} is a local counterexample for $\mathbf{A}(v\mathbf{V}\neg w)$. This proves that $\bigvee_{v \in V} \psi_v$ is false in $\pi(i)$, and that π^i is a local counterpath for each $\mathbf{AXA}(v\mathbf{V}\neg w)$ where $w \in V\setminus\{v\}$. Thus, π is a counterpath for ϕ in M.

($\Leftarrow$) Suppose that ϕ has a counterpath π in M. We show that the prefix $\pi(0), ..., \pi(n-1)$ of π is a Hamiltonian circuit of G. Let $v \in V$ be the node such that $\pi(0)=v$. Then, π is a counterpath for the formula ψ_v from above. This implies that π is a counterpath for the formula $\mathbf{AXA}(v\mathbf{V}\neg w)$, for each $w \in V\setminus\{v\}$. Thus, π^1 is a local counterpath for $\mathbf{A}(v\mathbf{V}\neg w)$. Hence, w must occur in π, and v must be false in each state $\pi(i)$ where $1 \leqslant i<k_w$ and $\pi(k_w)$ is the first occurrence of w in π. Consequently, $\pi(n)$ is the first possible position for a second occurrence of v in π.

Now consider $w_i=\pi(i)$, where $i>0$. By similar arguments, we obtain that each $w \in V\setminus\{w_i\}$ occurs in π^i, and that w must occur in π^i before any possible further occurrence of w_i after $\pi^i(0)=\pi(i)$. It follows that $\pi(0), \pi(1), ..., \pi(n-1)$ are all pairwise different, and that $\pi(n) = \pi(0)$ holds. This means that $\pi(0), ..., \pi(n-1)$ is a Hamiltonian circuit in G, and completes the proof of the claim.

Since M and ϕ are constructible in polynomial time from G, the result is proved. ∎

4.3. ACTL *Templates*

In the light of the previous results, we look into *structural properties* of formulas which guarantee the existence of a linear counterexample whenever a formula does not hold in a structure. This leads us to consider *templates* of ACTL formulas—formulas, in which the particular atomic propositions are meaningless, i.e., they can be substituted by arbitrary pure state formulas. Intuitively, a template expresses the structure of a formula in terms of linear-time and branching time operators. A pure state formula always has a linear counterexample (given by a single state); however, the application of these operators and Boolean connectives might destroy this property.

In the following, we shall identify the class of templates which are linear, i.e., each instantiation γ of a template $\gamma^\star$ obtained by filling in pure state formulas, has always a linear counterexample if γ is not true. As it turns out, this class is decidable, and in fact efficiently recognizable.

More formally, templates are defined as follows.

Definition 4.4 (template). A *template* $\gamma^\star$ is an ACTL formula over "$\star$" as single atomic proposition. The template of an ACTL formula γ, denoted $\gamma^\star$, is the template obtained by uniformly substituting "$\star$" for all atomic propositions in γ.[3]

Observe that for any ACTL formula γ, its template $\gamma^\star$ is unique. As with ordinary formulas, we shall often omit or introduce parentheses as usual.

Example 4.2. The template of $\gamma = \mathbf{A}(a\mathbf{V}\mathbf{AX}(b \wedge c))$ is $\gamma^\star = \mathbf{A}(\star\mathbf{V}\mathbf{AX}(\star \wedge \star))$, and the template of $\phi = \mathbf{A}((b \vee \neg c)\,\mathbf{U}\,a) \wedge \mathbf{AX}(c \wedge a))$ is $\phi\star = \mathbf{A}((\star \vee \neg\star)\mathbf{U}\star) \wedge \mathbf{AX}(\star \wedge \star))$.

Definition 4.5 ($\mathbf{T}^\star, \mathscr{PSF}$). We denote by $\mathbf{T}^\star$ the set of all ACTL templates and by $\mathscr{PSF} \subseteq \mathbf{T}^\star$ the set of pure state formulas on the atomic proposition $\star$.

Instantiations of templates are defined as follows.

Definition 4.6 (instantiation). An ACTL formula ϕ over atoms AP, where $\star \notin AP$, is an *instantiation* of a template $\gamma^\star \in \mathbf{T}^\star$, if ϕ results by substituting each occurrence of $\star$ in $\gamma^\star$ with a (possibly different) pure state formula over AP.

Example 4.3. An instantiation of $\mathbf{A}(\star\mathbf{V}(\neg\star \vee \mathbf{A}(\star\mathbf{U}\star))$ is $\mathbf{A}(\mathit{false}\mathbf{V}(\neg \mathit{req} \vee \mathbf{A}(\mathit{true}\mathbf{U}\mathit{ack})))$, which expresses that a request is always finally acknowledged (see [5] for this formula). Among the instantiations of $\mathbf{A}((\star \vee \neg\star)\mathbf{U}\star) \wedge \mathbf{AX}(\star \wedge \star))$ are $\mathbf{A}((b \vee \neg c)\,\mathbf{U}\,(b \wedge a)) \wedge \mathbf{AX}(c \wedge \neg a))$ and $\mathbf{A}((a \vee \neg a)\,\mathbf{U}\,a) \wedge \mathbf{AX}(a \wedge \neg a))$, i.e., $\mathbf{A}(\mathit{true}\,\mathbf{U}\,a) \wedge \mathbf{AX}(\mathit{false}))$.

Linear templates are now defined by abstraction from c-linear formulas.

Definition 4.7 (c-linear template). A template $\gamma^\star$ is *c-linear*, if each instantiation ϕ of $\gamma^\star$ is c-linear.

[3] Alternatively, we could define that maximal pure state formulas in γ are replaced by $\star$, rather than atoms. However, the definition of **LIN** and the BNF grammar in Table 1 would become more complex, while the main results are not affected.

Examples of c-linear templates are given in Example 4.4 below.

We next define a subset $\mathbf{LIN} \subseteq \mathbf{T}^\star$ of templates in terms of the least fixpoint of a continuous operator which is applied to a pair of sets of templates. The main effort in the rest of the paper will be the proof that this set **LIN** is precisely the set of all c-linear templates.

Definition 4.8 (operator Λ). The operator $\Lambda: 2^{\mathbf{T}^\star} \times 2^{\mathbf{T}^\star} \to 2^{\mathbf{T}^\star} \times 2^{\mathbf{T}^\star}$ is defined as follows:

$$\Lambda(S_1, S_2) = (S'_1, S'_2),$$

where

$$S'_1 = \mathscr{PSF} \cup S_1 \wedge S_1 \cup S_1 \vee \mathscr{PSF} \cup \mathscr{PSF} \vee S_1 \cup \mathbf{AX}(S_1) \cup \mathbf{AV}(\mathscr{PSF}, S_1) \cup S_2$$

$$S'_2 = \mathbf{AU}(S_1, \mathscr{PSF}) \cup \mathbf{AU}(\mathscr{PSF}, S_2) \cup S_2 \vee \mathscr{PSF} \cup \mathscr{PSF} \vee S_2$$

Obviously, Λ is a continuous operator on a complete lattice, and hence by Kleene's Theorem, the least fixpoint $\Lambda^\infty = (S_1^\infty, S_2^\infty)$ exists and is the limit of the sequence $\Lambda^0 = (\emptyset, \emptyset)$, $\Lambda^{i+1} = \Lambda(\Lambda^i)$, $i \geqslant 0$.

Definition 4.9 (**LIN**). We define $\mathbf{LIN} = S_1^\infty$ as the first component of the least fixpoint $\Lambda^\infty = (S_1^\infty, S_2^\infty)$ of Λ.

Example 4.4. As easily checked, the sample templates in Section 1 generated by the grammar in Table 1 are in **LIN**. In fact, it is easy to see that **LIN** coincides with the language generated by that grammar. Further templates belonging to **LIN** are: $\mathbf{AXAX}(\bigstar)$, $\mathbf{AXA}(\bigstar\mathbf{U}(\bigstar \vee \neg\bigstar))$, $\mathbf{A}((\mathbf{AX}(\bigstar))\mathbf{U}(\bigstar \wedge \bigstar))$, $\mathbf{A}((\neg(\bigstar \vee \bigstar))\mathbf{VAX}(\bigstar))$, $\mathbf{A}(\bigstar\mathbf{V}((\neg\bigstar) \vee \mathbf{A}(\bigstar\mathbf{U}\bigstar)))$, and $\mathbf{A}(\mathbf{A}(\bigstar\mathbf{U}\bigstar)\mathbf{U}\bigstar)$. On the other hand, the templates $\mathbf{A}((\mathbf{AX}(\bigstar))\mathbf{V}\bigstar)$ and $\mathbf{A}(\bigstar\mathbf{U}(\bigstar \wedge \mathbf{AX}(\bigstar)))$ are not in **LIN**.

The first of the main results of this paper can now be stated as follows.

Theorem 4.3. *Let $\gamma^\star \in \mathbf{T}^\star$. Then, $\gamma^\star$ is c-linear if and only if $\gamma^\star \in \mathbf{LIN}$.*

From this result and the inductive definition of **LIN**, we easily obtain the following corollary concerning the recognition of linear templates; observe that membership of a template in **LIN** can be checked in a single bottom up pass of the formula tree, in which each step is unambiguous.

Corollary 4.4. *Given a template $\gamma^\star \in \mathbf{T}^\star$, deciding whether $\gamma^\star$ is c-linear is possible in $O(|\gamma^\star|)$ time, where $|\gamma^\star|$ is the length of $\gamma^\star$.*

The proof of Theorem 4.3 is rather technical, and involves detailed case distinctions. It is given in Sections 5 (if-part) and 6 (only-if part).

5. TEMPLATES IN LIN ARE c-LINEAR

In this section, we prove in Theorem 5.1 that all instances of templates in **LIN** are c-linear. The proof proceeds along the inductive definition of **LIN**.

It appears that using an inductive inductive argument, we can establish that any next-time formula $\mathbf{AX}\phi_1$ is c-linear provided that ϕ_1 is, and similarly that nesting any c-linear formula ϕ_1 (resp., ϕ_2) into the left argument of an until $\mathbf{A}(\phi_1\mathbf{U}\phi_2)$ (resp., right argument ϕ_2 of an unless $\mathbf{A}(\phi_1\mathbf{V}\phi_2)$) results in a c-linear formula, if ϕ_2 (resp., ϕ_1) is a pure state formula. However, it appears that c-linearity is not strong enough to allow the induction step go through smoothly for all templates, and in particular for nesting non-pure state formula into the right argument of an until. We can remedy this problem by revealing that a strengthened version of c-linearity is satisfied by some of the templates, and exploit that this stronger property can be established in the induction step comparatively easy.

Definition 5.1 (strongly c-linear). An ACTL formula ϕ is *strongly c-linear*, if ϕ is c-deterministic and the following two conditions hold for any Kripke structure M:

1. if Π is a linear ℓ-counterexample for ϕ in M, then every path π of form $\pi = s_0, ..., s_k, \mu(\Pi)$ in M such that $s_0 \in S_0(M)$ and ϕ has ℓ-counterexamples at $s_0, ..., s_k$ is a counterpath of ϕ; and

2. if π is a path in M such that $\pi(0) \in S_0(M)$ and every $\pi(i)$, $i \geqslant 0$, is the origin of some ℓ-counterexample for ϕ in M, then π is a counterpath for ϕ in M.

A template $\gamma^\star$ is *strongly c-linear*, if every instantiation ϕ of $\gamma^\star$ is strongly c-linear.

Example 5.1. The formula $\phi = \mathbf{A}(a\mathbf{U}b)$ is strongly c-linear: a local counterexample Π for ϕ is a path π, and at the state $\pi(0)$, the atom b is false. By adding a prefix $s_0, ..., s_{k-1}$ of states to π such that b is false in each state s_i, we clearly obtain a path $\pi' = s_0, ..., s_{k-1}, \pi$ witnessing that $a\mathbf{U}b$ is false, i.e., π' is a counterpath for ϕ. Thus, item 1 of strong c-linearity is satisfied. Also item 2 is satisfied: b must be false at the origin of any local counterexample of ϕ; thus, if π is a path as described in item 2, b is false at each state $\pi(i)$. This means that π is a counterexample (and thus a counterpath) for ϕ.

It is easy to see that this holds if the atoms a and b are replaced by arbitrary pure state formulas; thus, the templates $\mathbf{A}(\star\mathbf{U}\star)$ and all templates in $\mathbf{AU}(\mathcal{PSF}, \mathcal{PSF})$ are strongly c-linear.

On the other hand, the formula $\phi = \mathbf{A}(a\mathbf{V}b)$, even if it is c-linear (as we shall see below), is not strongly c-linear, since it fails to satisfy item 2 of the definition. Indeed, consider a path π where each $\pi(i)$ is the origin of a local counterexample for ϕ, in which a is false and b is true. Then, b is true in each state of π. However, a counterexample for ϕ must involve a state at which b is false. Thus, π is not a counterpath for ϕ and item 2 fails. It is easy to see from this that no template in $\mathbf{AV}(\mathcal{PSF}, \mathcal{PSF})$ is strongly c-linear. Similarly, it is easy to see that $\mathbf{AX}a$ is not strongly c-linear (both item 1 and 2 may fail), and that no template in $\mathbf{AX}(\mathcal{PSF})$ is strongly c-linear.

As for more complex formulas, e.g., the templates $\mathbf{A}(\star\mathbf{U}(\star\mathbf{U}\star))$ and $\mathbf{A}(\star\mathbf{U}\star) \vee \star$ are strongly c-linear. This will be formally proven below.

In Theorem 5.1 we now show that the templates in the class **LIN** are sound with respect to the property of c-linearity, i.e., each template in this class is c-linear. In fact, in the proof of the result we establish a little more, namely that all templates in the subset $S_2^\infty \subseteq \mathbf{LIN}$ are strongly c-linear.

Strong c-linearity helps us in building a counterpath for an until formula $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where γ_2 is another until formula $\mathbf{A}(\gamma_{2,1} \mathbf{U} \gamma_{2,2})$, inductively from a counterpath for γ_2. As γ_2 is strongly c-linear, we obtain by item 1 of Definition 5.1 a counterpath for γ_2 if we can reach from some initial state s_0 over a sequence of states in which γ_2 fails some local counterpath π for γ_2. Now if Π is an arbitrary counterexample for γ which involves the failure of $\gamma_1 \vee \gamma_2$ at some point $\Pi(k)$, then in case γ_1 is a pure state formula we can simply take as this sequence $s_0 = or(\Pi(0))$, $or(\Pi(1))$, ..., $or(\Pi(k-1))$ and for π we take *any* counterpath for γ_2 that starts in $or(\Pi(k))$—such a counterpath will exist and its origin will disprove γ_1; note that the latter will not be true in general if a counterexample for γ_1 involves a path. In case Π shows failure of γ_2 at each stage, then we are guaranteed by item 2 of Definition 5.1 that $\mu(\Pi)$ is a counterpath for γ. Intuitively, $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$ inherits strong c-linearity from γ_2, as any counterexample for γ involves an initial (or infinite) sequence of counterexamples for γ_2 and γ_1 needs no path for refutation. This is similar for any disjunction $\gamma_2 = \phi_1 \vee \phi_2$ of an until formula ϕ_1 and a pure state formula ϕ_2, but fails for every such conjunction $\phi_1 \wedge \phi_2$: failure of ϕ_2 might release failure of ϕ_1 at a state s_i in a prefix $s_0, ..., s_k$ to a counterpath for γ_2, and prevent that some s_j where $j < i$ has a counterexample in the resulting path.

We illustrate this by the following example. Consider the formula $\gamma = \mathbf{A}(a\mathbf{U}\mathbf{A}(b\mathbf{U}c))$, and let Π be a counterexample for γ in a structure M. Suppose that Π shows failure of $a \vee \mathbf{A}(b\mathbf{U}c)$ at some stage $k \geqslant 0$ and that $\Pi(i)$ is a counterexample for $\mathbf{A}(b\mathbf{U}c)$ for all $0 \leqslant i < k$. Then, a is false in the initial stage of $\Pi(k)$, which is a path such that either $b \vee c$ is false at some stage j and c is false at all previous stages, or c is false at every stage. Since $\Pi(i)$ is for every $0 \leqslant i < k$ a counterexample for $\mathbf{A}(b\mathbf{U}c)$, the formula c must be false at the initial stage of $\Pi(i)$. Now the path π obtained by prefixing $\Pi(k)$ with $or(\Pi(0))$, ..., $or(\Pi(k-1))$ is a counterpath for γ: indeed, each suffix π^i for $0 \leqslant i < k$ is a counterpath for $\mathbf{A}(b\mathbf{U}c)$ (as predicted by item 1 of Definition 5.1) and π^k is a counterpath for $a \vee \mathbf{A}(b\mathbf{U}c)$. Otherwise, suppose Π is such that $\Pi(i)$ for $i \geqslant 0$ is a counterexample for $\mathbf{A}(b\mathbf{U}c)$. Clearly, the right argument c is false at the origin of $\Pi(i)$. Thus, $\mathbf{A}(b\mathbf{U}c)$ is false along the path $\pi = [or(\Pi(0)), or(\Pi(1)), ...] = \mu(\Pi)$ (as predicted by item 2 of Definition 5.1) because c never becomes true, which means that π is a counterpath for γ. Thus, in both cases, γ has a counterpath in M. However, no counterpath for $\gamma = \mathbf{A}(a\mathrm{U}(\mathbf{A}(b\mathbf{U}c) \wedge b))$ may be obtained from a counterexample Π for γ: e.g., $\Pi(0)$ may be a counterexample for $\phi_1 = \mathbf{A}(b\mathbf{U}c)$ and $\Pi(1)$ for both $\phi_2 = d$ and $\gamma_1 = a$ (thus for $a \vee (\phi_1 \wedge \phi_2)$), while ϕ_2 and ϕ_1 are true at $or(\Pi(0))$ and $or(\Pi(1))$, respectively. It is then impossible to build a counterpath for γ by prefixing a path starting at $or(\Pi(1))$ with $or(\Pi(0))$ (cf. also proof of Theorem 6.7, case 4.4).

Let us now see whether we can obtain a similar result for an unless formula $\mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$ by swapping, like above, the left and right argument in a until. It appears that it is not possible to nest anything else than a pure state formula into γ_1 without losing c-linearity. Would we do so, then even strong c-linearity would not ensure that the formula is c-linear. Recall that a counterexample for $\mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$ is a multi-path $\Pi = [\Pi(0), \Pi(1), ...]$ such that $\Pi(0), ..., \Pi(k-1)$ prove the falsity of γ_1 and

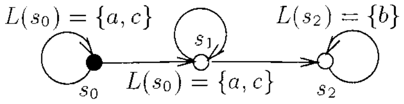

FIG. 5. Transition graph representing structure M (initial state s_0).

$\Pi(k)$ the falsity of γ_2. Trying to construct from Π a linear counterexample $\bar{\Pi}$ for $\mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$, we have to replace each $\Pi(i)$, $0 \leqslant i \leqslant k$, with a suitable linear counterexample $\bar{\Pi}(i)$. We can do so easily for all $i < k$: Since γ_1 is strongly c-linear, for any linear counterexample $\bar{\Pi}(k-1)$ for γ_1 we can find appropriate $\bar{\Pi}(0), ...,$ $\bar{\Pi}(k-2)$ by exploiting the property in item 1 of Definition 5.1. However, it may happen that every possible $\bar{\Pi}(k-1)$ misses some state from $\Pi(k)$ which is necessary to refute γ_2; thus, a linear counterexample $\bar{\Pi}$ can not be built.

For example, consider $\gamma = \mathbf{A}(\mathbf{A}(a\mathbf{U}b)\mathbf{V}c)$, i.e., nesting of $\mathbf{A}(a\mathbf{U}b)$ (which is strongly c-linear), and the structure M corresponding to the transition graph in Fig. 5. Observe that $M \not\models \gamma$, which is witnessed by the multi-path $\Pi = [[s_0, s_0, s_0, ...],$ $[s_1, s_1, s_1, ...], s_2, s_2, ...]$. Indeed, the paths $\Pi(0)$ and $\Pi(1)$ are counterexamples for $\mathbf{A}(a\mathbf{U}b)$, as b is always false along them, and $\Pi(2) = s_2$ is a counterexample for c (i.e., $k = 2$). Clearly this multi-path is not linear. In this case, strong c-linearity of the formula $\mathbf{A}(a\mathbf{U}b)$ does not help us to construct a counterpath for γ from Π. While the path $\pi = [s_0, s_1, s_1, ...]$ obtained by prefixing $\Pi(1) = [s_1, s_1, ...]$ with $s_0 = or(\Pi(0))$ is a counterpath for $\mathbf{A}(a\mathbf{U}b)$, it is not a counterpath for γ, because c is always true along it. Observe also that γ has no counterpaths in M at all. Indeed, any counterpath π must contain as suffix the path $[s_2, s_2, ...]$, since π must witness the falsity of c. On the other hand, clearly no path in M with suffix $[s_2, s_2, ...]$ is a counterpath for $\mathbf{A}(a\mathbf{U}b)$.

THEOREM 5.1. Every template in **LIN** is c-linear.

Proof. We establish the result proving by induction on the stages $\Lambda^i = (S_1^i, S_2^i)$, $i \geqslant 0$, that every template $\gamma^\star \in S_1^i$ is c-linear and every template $\gamma^\star \in S_2^i$ is strongly c-linear.

(Basis) The case $i = 0$ is trivial, since $S_1^0 = S_2^0 = \varnothing$.

(Induction) Consider $i+1$ and assume the statement holds for i. Let $\gamma^\star$ be any template such that $\gamma^\star \in S_1^{i+1} \setminus S_1^i$ (resp., $\gamma^\star \in S_2^{i+1} \setminus S_2^i$).

To complete the proof it suffices to show that $\gamma^\star$ is c-linear (resp., strongly c-linear), i.e. each instantiation ϕ of $\gamma^\star$ is c-linear (resp., strongly c-linear).

Let M be any Kripke structure such that $M \not\models \phi$. Then, we have to prove that a linear counterexample for ϕ exists in M. From the definition of Λ, the following cases for $\gamma^\star$ are possible.

- $\gamma^\star \in \mathcal{PSF} \subseteq S_1^{i+1}$. (In this case, $i = 0$.) Each counterexample of ϕ in M is finite, and thus linear.

- $\gamma^\star \in S_1^i \wedge S_1^i \subseteq S_1^{i+1}$. Thus, $\phi = \gamma_1 \wedge \gamma_2$, where both γ_1 and γ_2 are c-linear by induction hypothesis. Since $M \not\models \phi$, either $M \not\models \gamma_1$ or $M \not\models \gamma_2$. In both cases, the statement follows from the induction hypothesis.

• $\gamma^\star \in S_1^i \vee \mathscr{PSF} \cup \mathscr{PSF} \vee S_1^i \subseteq S_1^{i+1}$. Then, $\phi = \gamma_1 \vee \gamma_2$. Assume γ_2 is a pure state formula and γ_1 is an instantiation of a template in S_1^i; the other case (vice versa) is similar. By the induction hypothesis, γ_1 is c-linear.

Since $M \not\models \phi$, hence $M, s_0 \not\models \gamma_1$ and $M, s_0 \not\models \gamma_2$ for some initial state s_0. Moreover, since γ_1 is c-linear, it admits a linear counterexample Π_{γ_1} also in M_{s_0}.[4] Clearly, $or(\Pi_{\gamma_1}) = s_0$ and Π_{γ_1} is a counterexample for γ_1 in M too. Hence the linear multi-path $\Pi_{\gamma_1} * s_0 = \Pi_{\gamma_1}$ is a counterexample for $\gamma_1 \vee \gamma_2$ in M. Thus, ϕ is c-linear.

• $\gamma^\star \in \mathbf{AX}(S_1^i) \subseteq S_1^{i+1}$. Consequently, ϕ is of shape $\mathbf{AX}(\gamma_1)$, where γ_1 is an instantiation of a template in S_1^i. Since $M \not\models \phi$, there must exist a path π such that $\pi(0) \in S_0(M)$ and $M, \pi(1) \not\models \gamma_1$. By the induction hypothesis, γ_1 is c-linear. Thus, γ_1 has a linear counterexample, say Π_{γ_1}, also in $M_{\pi(1)}$. Consider now the multi-path Π defined as follows: $\Pi(0) = \pi(0)$, $\Pi(1) = \Pi_{\gamma_1}$, and $\Pi(i) = \mu(\Pi_{\gamma_1})(i-1)$ if γ_1 is not a pure state formula, and $\Pi(i) = \pi(i)$ otherwise, for each $i > 1$. Clearly, $\Pi(1)$ is a ℓ-counterexample for γ_1 in M. Hence, Π is a counterexample for ϕ; clearly, it is linear.

• $\gamma^\star \in \mathbf{AV}(\mathscr{PSF}, S_1^i) \subseteq S_1^{i+1}$. Then $\phi = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$, where γ_1 is a pure state formula and γ_2 is c-linear by the induction hypothesis. Since $M \not\models \phi$, there exists a path π and a $k \geqslant 0$ with $\pi(0) \in S_0(M)$ such that $M, \pi(k) \not\models \gamma_2$ and $M, \pi(i) \not\models \gamma_1$ for every $0 \leqslant i < k$. Since γ_2 is c-linear, by the induction hypothesis there exists a linear counterexample Π_{γ_2} for γ_2 in $M_{\pi(k)}$. Hence, the multi-path Π such that $\Pi(i) = \pi(i)$, for each $0 \leqslant i < k$, $\Pi(k) = \Pi_{\gamma_2}$, and $\Pi(i+k) = \mu(\Pi_{\gamma_2})(i)$, if γ_2 is not a pure state formula, and $\Pi(i+k) = \pi(i+k)$ otherwise, for $i \geqslant 1$, is a counterexample for ϕ in M. Since Π is linear, it follows that ϕ is c-linear.

• $\gamma^\star \in S_2^i \subseteq S_1^{i+1}$. By the induction hypothesis.

• $\gamma^\star \in \mathbf{AU}(S_1^i, \mathscr{PSF}) \subseteq S_2^{i+1}$. We show first that ϕ is c-linear. ϕ is of the form $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where γ_1 is c-linear by the induction hypothesis and γ_2 is a pure state formula. Let Π be a counterexample for ϕ in M. By definition of counterexample, Π is such that either

7.1. $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$, or

7.2. there exists a $k \geqslant 0$ such that $\Pi(k)$ is a counterexample for $\gamma_1 \vee \gamma_2$, $\Pi(i)$ is a counterexample for γ_2 (and thus it is a state), for each $0 \leqslant i < k$ and $\Pi(j)$ is a state, for each $j > k$.

In case 7.1, since γ_2 is a pure state formula, $\Pi(i)$ is a state, for each $i > 0$, and, hence, it is a linear counterexample. Consider now case 7.2. As shown above, each template in $S_1^i \vee \mathscr{PSF}$, is c-linear, and thus $\gamma_1 \vee \gamma_2$ is c-linear. Hence, $\gamma_1 \vee \gamma_2$ has a linear counterexample also in $M_{\mu(\Pi)(k)}$. Let $\Pi_{\gamma_1 \vee \gamma_2}$ be any such linear counterexample. Consider now the multi-path Π_ϕ defined as follows: $\Pi_\phi(i) = \Pi(i)$ for each $0 \leqslant i < k$, $\Pi_\phi(k) = \Pi_{\gamma_1 \vee \gamma_2}$, $\Pi_\phi(j) = \mu(\Pi_{\gamma_1 \vee \gamma_2})(j-k)$, for $j > k$. Clearly, $\Pi_\phi(k)$ is

[4] Recall that, for any structure M and state $s \in S(M)$, M_s denotes the structure resulting from M with the set of initial states redefined to $\{s\}$.

a counterexample for $\gamma_1 \vee \gamma_2$ in M. Hence, Π_ϕ is a counterexample for ϕ in M. Further, as can be easily checked, Π_ϕ is linear.

After proving that ϕ is c-linear, we prove that ϕ satisfies item 1 of Definition 5.1. Consider a path $\pi = s_0, ..., s_k, \mu(\Pi)$, as there, where Π is a linear ℓ-counterexample for ϕ in M. Recall that $\phi = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where γ_1 is, by the induction hypothesis, c-linear and γ_2 is a pure state formula. $M_{s_i} \not\models \phi$ implies that γ_2 is false at s_i, for each $i = 0, ..., k$. Since Π is a linear counterexample for ϕ in $M_{or(\Pi)}$, either

(α) there exists a $j \geqslant 0$ such that $\Pi(j)$ is a counterexample for $\gamma_1 \vee \gamma_2$ and $\Pi(i)$, for each $0 \leqslant i < j$, is a ℓ-counterexample for γ_2 (and thus a state), or

(β) $\Pi(i)$, is a ℓ-counterexample for γ_2 for each $i \geqslant 0$ (hence Π is a path).

In either case, the multi-path $\bar{\Pi} = [s_0, ..., s_k, \Pi(0), \Pi(1), ...]$ is a counterexample for ϕ in M (recall that $s_0 \in S_0(M)$), which is clearly linear. Since $\pi = \mu(\bar{\Pi})$ item 1 of Definition 5.1 is satisfied.

To show that ϕ satisfies also item 2 of Definition 5.1, consider any path π such that $\pi(0) \in S_0(M)$ and $\pi(i)$ is the origin of some ℓ-counterexample for ϕ in M, for each $i \geqslant 0$. Thus, γ_2 is false in each state $\pi(i)$, for $i \geqslant 0$. Hence, π is a counterpath for ϕ in M.

- $\gamma^\star \in \mathbf{AU}(\mathscr{PSF}, S_2^i) \subseteq S_2^{i+1}$. Then ϕ is of the shape $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where γ_1 is a pure state formula and γ_2 is strongly c-linear by the induction hypothesis. We have to prove that also ϕ is strongly c-linear. We first show that ϕ is c-linear. Consider thus a counterexample Π for ϕ. Then, either

8.1. there exists a $k \geqslant 0$ such that $\Pi(k)$ is a counterexample for $\gamma_1 \vee \gamma_2$ and $\Pi(i)$ is a counterexample for γ_2, for each $0 \leqslant i < k$, or

8.2. $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$.

In the case (8.1), by definition of counterexample $M_{or(\Pi(i))} \not\models \gamma_2$, for each $0 \leqslant i < k$. Consider now any linear counterexample Π_{γ_2} for γ_2 in $M_{or(\Pi(k))}$. Such a counterexample exists, since γ_2 is strongly c-linear (thus c-linear). Hence, by item 1 of Definition 5.1, it follows that for every path $\pi_j = [or((\Pi)(j)), ..., or((\Pi)(k-1)), \mu(\Pi_{\gamma_2})(0), \mu(\Pi_{\gamma_2})(1), ...]$, for all $0 \leqslant j \leqslant k$, there exists a linear counterexample Π_j for γ_2 in $M_{or(\Pi(j))}$ such that $\mu(\Pi_j) = \pi_j$. Hence, the multi-path $\bar{\Pi}$ such that $\bar{\Pi}(i) = \Pi_i$, for $0 \leqslant i < k$, $\bar{\Pi}(k) = \Pi_{\gamma_2}$, and $\bar{\Pi}(i+k) = \mu(\Pi_{\gamma_2})(i)$, for $i > 0$, is a counterexample for ϕ. Moreover, as can be easily verified, each Π_j, for $0 \leqslant j < k$, is linear.

In the case (8.2), by definition of counterexample $M_{or(\Pi(i))} \not\models \gamma_2$, for each $i \geqslant 0$. Since γ_2 is strongly c-linear, it satisfies item 2 of Definition 5.1. Thus, each suffix $\mu(\Pi)^j$ is a counterpath for γ_2. Hence, for any linear counterexamples of $\bar{\Pi}_i$ of γ_2 such that $\mu(\bar{\Pi}_i) = \mu(\Pi)^i$, $i \geqslant 0$, the linear multi-path $[\bar{\Pi}_0, \bar{\Pi}_1, ..., \bar{\Pi}_i, ...]$ is a linear counterexample for ϕ.

After proving that ϕ is c-linear, it remains to prove that ϕ satisfies items 1 and 2 of Definition 5.1. Let $\pi = s_0, s_1, ..., s_k, \mu(\Pi)$ be a path as in item 1 for a linear ℓ-counterexample Π of ϕ in M. Recall that $\phi = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where γ_1 is a pure state formula and γ_2 is, by the induction hypothesis, strongly c-linear. Since s_i is origin of some ℓ-counterexample for ϕ in M, it follows $M_{s_i} \not\models \gamma_2$, for each $0 \leqslant i \leqslant k$. Furthermore, since Π is a linear counterexample for ϕ, either

(α) there exists a $j \geqslant 0$ such that $\Pi(j)$ is a counterexample for $\gamma_1 \vee \gamma_2$ and $\Pi(i)$ is a counterexample for γ_2, for each $0 \leqslant i < j$, or

(β) $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$.

In any case, γ_2 has a linear ℓ-counterexample $\bar{\Pi}$ at $or(\Pi)$ such that $\mu(\bar{\Pi}) = \mu(\Pi)$. Since γ_2 is strongly c-linear, item 1 of Definition 5.1 implies that for each $i = 0, ..., k$ a linear ℓ-counterexample Π_i for γ_2 exists at s_i such that $\mu(\Pi_i) = \pi^i$. Hence, the multi-path $\Pi' = [\Pi_0, ..., \Pi_k, \bar{\Pi}(0), \bar{\Pi}(1), ...]$ is a linear counterexample for ϕ in M. Since $\mu(\Pi') = \pi$, π is a counterpath for ϕ in M; thus, item 1 is satisfied.

To show that ϕ satisfies also item 2 of Definition 5.1, let π be a path in M such that $\pi(0) \in S_0(M)$ and each $\pi(i)$ is origin of a ℓ-counterexample for ϕ in M, $i \geqslant 0$. Then, each $\pi(0)$ must be the origin of a ℓ-counterexample for γ_2. Since γ_2 is strongly c-linear, it follows from item 2 of Definition 5.1 that each suffix π^i of π, $i \geqslant 0$, is a counterpath for γ_2 in M, i.e., a corresponding linear ℓ-counterexample Π_i for γ_2 exists in M at $\pi(i)$. Thus, $\Pi = [\Pi_0, \Pi_1, ...]$ is a linear counterexample for ϕ in M such that $\pi = \mu(\Pi)$. This means π is a counterpath for ϕ in M, and item 2 of Definition 5.1 is satisfied.

- $\gamma^\star \in S_2^i \vee \mathscr{PSF} \cup \mathscr{PSF} \vee S_2^i \subseteq S_2^{i+1}$. The proof that $\gamma^\star$ is c-linear is analogous to the case $\gamma^\star \in S_1^i \vee \mathscr{PSF} \cup \mathscr{PSF} \vee S_1^i$ above. The verification of points 1 and 2 in Definition 5.1 is straightforward. ∎

5.1. Computing a Counterpath for **LIN**-*Instances*

In Section 4, we have shown that deciding whether an arbitrary formula ϕ has a counterpath on a given structure M is intractable in general, and so is computing a counterpath. Since instances of **LIN**-templates always have a counterpath if they are false in M, the question whether there is an (efficient) procedure for computing any counterpath is natural. Note that existence of a counterpath does not a priori mean that computing a counterpath is easy; this could still be a difficult problem.

Our second main result shows that this is not the case. Let for any finite path $P = s_0, s_1, ..., s_k$ in a structure M denote $|P|$ the length of P ($= k+1$), and let for any formula γ denote $d_{\mathbf{A}}(\gamma)$ the **A**-nesting depth of γ (where $d_{\mathbf{A}}(\gamma) = 0$ for every pure state formula γ).

THEOREM 5.2. *Let γ be such that $\gamma^\star \in$ **LIN**. If $M \not\models \gamma$, then γ has a counterpath in M which is either a single state (if $\gamma^\star \in \mathscr{PSF}$), or representable as P, C where P is a finite path (prefix) and C a cycle in M such that $|P| + |C| \leqslant d_{\mathbf{A}}(\gamma)\,|S(M)|$. Moreover, given γ and M, such P and C can be computed in polynomial time.*

Proof. The first part (existence of a representation P, C as described) is shown following the induction in the proof of Theorem 5.1. For each instance ϕ of a template $\gamma^\star \in S_1^i \cup S_2^i$, we can construct the desired representation P, C from the main path of the linear counterexample constructed in the proof there, exploiting that linear counterexamples Π' used in the constructions have representations P', C' as described. We omit repeating all these constructions in detail, and focus here on the relevant facts that establish P, C:

1. In cases where ϕ is of the form $\phi_1 \vee \phi_2$, $\phi_1 \wedge \phi_2$, a counterpath for ϕ is immediately obtained by the induction hypothesis.

2. In cases where ϕ is of the form $\mathbf{AX}\phi_1$, $\mathbf{A}(\phi_1 \mathbf{V} \phi_2)$, and in some cases of $\mathbf{A}(\phi_1 \mathbf{U} \phi_2)$, the linear counterexample Π constructed for ϕ is of the form $[\Pi(0), ..., \Pi(k), \Pi(k+1), ...]$ where $\Pi(0), ..., \Pi(k-1)$ are states except if $\phi^\star \in \mathbf{AU}(\mathscr{PSF}, \mathbf{LIN} \backslash \mathscr{PSF})$, $\Pi(k)$ is a linear counterexample for a formula ψ' such that $d_\mathbf{A}(\psi') < d_\mathbf{A}(\phi)$, and all $\Pi(j)$ are states, $j > k$. Two subcases arise, depending on the formula ψ':

2.1. $d_\mathbf{A}(\psi') = 0$, i.e., $\psi'^\star \in \mathscr{PSF}$. Then, Π is a simple path in M, and the states $\Pi(j)$, $j > k$, in Π are meaningless (i.e., the suffix $[\Pi(k), \Pi(k+1) \cdots]$ can be replaced by any infinite path starting at $\Pi(k)$). Thus, a counterpath for ϕ can be represented by P, C such that $|P| + |C| \leqslant |S(M)| \leqslant d_\mathbf{A}(\phi) |S(M)|$.

2.2. $d_\mathbf{A}(\psi') > 0$. Then, ψ' can be assumed to have a counterpath P', C' as in the induction hypothesis, and P, C is given by $s_0, ..., s_{k-1}, P', C'$, where $s_i = or(\Pi(i))$, for $i = 0, ..., k-1$. For a minimal k, it holds that $k \leqslant |S(M)|$, and we obtain

$$|P| + |C| = k + |P'| + |C'| \leqslant |S(M)| + d_\mathbf{A}(\psi')\ |S(M)| \leqslant d_\mathbf{A}(\phi)\ |S(M)|.$$

3. In the case where $\phi = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, a linear counterexample Π may be constructed such that each $\Pi(i)$ is a counterexample for γ_2. In the case where $\gamma_2^\star \in \mathscr{PSF}$, Π is a simple path in M, which can be replaced by a prefix-cycle pair P, C such that $|P| + |C| \leqslant |S(M)| \leqslant d_\mathbf{A}(\phi)\ |S(M)|$ (cf. 2.1); otherwise, if $\gamma_2^\star \in \mathbf{LIN} \backslash \mathscr{PSF}$, then P, C is given by P', C' representing $\mu(\Pi(0))$, and by the induction hypothesis $|P| + |C| = |P'| + |C'| \leqslant d_\mathbf{A}(\gamma_2)\ |S(M)| \leqslant d_\mathbf{A}(\phi)\ |S(M)|$.

This concludes the proof of the first part of the theorem. For computing P, C in polynomial time (second part of Theorem 5.2) we describe an algorithm which proceeds in two steps. Suppose that ϕ and M are given for input.

Step 1. Label each state $s \in S$ with the set

$$F(s) = \{\phi' \mid \phi' \text{is a subformula of } \phi \text{ such that } M, s \not\models \phi'\}.$$

It is well-known that this labeling is possible in polynomial time (in fact in $O(|\phi|(|S(M)| + |R(M)|)$ time) [3].

Step 2. Construct a counterpath for ϕ, which is either a single state or P, C representing an infinite path, using the following procedure:

Procedure COUNTERPATH

Input: Labeled graph $G = (S, R, F)$, **LIN** instance ϕ, state $s \in S$ s.t. $\phi \in F(s)$.

Output: s, if $\phi^\star \in \mathscr{PSF}$; otherwise, P, C representing a counterpath π for ϕ starting at s.

Execute COUNTERPATH(G, ϕ, s_0) for *some arbitrary* $s_0 \in S_0$ such that $\phi \in F(s_0)$, and return the result.

COUNTERPATH proceeds *top-down*, and constructs the output either directly, or by making a recursive call; thus, COUNTERPATH extends an initially empty prefix P_0 to $P_1 \leqslant P_2 \leqslant \cdots$ repeatedly until it is eventually completed with a cycle. In general, different choices exist for extending P_i to P_{i+1}. The crucial fact is that membership of $\phi\star$ in **LIN** guarantees a "don't care" nondeterminism, i.e., no backtracking is necessary. If P_i is properly extended to P_{i+1}, then it can be finally completed with a cycle.

We now describe how COUNTERPATH proceeds for $\phi^\star \notin \mathscr{PSF}$, depending on the structure of ϕ. We consider the different possible cases:

- $\phi = \gamma_1 \wedge \gamma_2$. Then, either $\gamma_1 \in F(s)$ or $\gamma_2 \in F(s)$ (or both). Call either COUNTERPATH(G, γ_1, s) or COUNTERPATH(G, γ_2, s), respectively, and return the result.

- $\phi = \gamma_1 \vee \gamma_2$. If $\gamma_1^\star \in \mathscr{PSF}$, then call COUNTERPATH$(G, \gamma_2, s)$; otherwise, call COUNTERPATH(G, γ_1, s). Return the result.

- $\phi = \mathbf{AX}(\gamma_1)$. Choose any s' such that $(s, s') \in R$ and $\gamma_1 \in F(s')$. If $\gamma_1^\star \notin \mathscr{PSF}$, then call COUNTERPATH$(G, \gamma_1, s')$ and return the result; otherwise, complete the path s, s' to an arbitrary prefix-cycle path P, C (where P may be void) containing at most $|S(M)|$ states.

- $\phi = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$. Determine any node s' reachable by a (possible empty) path $s = s_0, s_1, ..., s_k = s'$ in R such that $\gamma_1 \in F(s_i)$, for all $i = 0, ..., k-1$ and $\gamma_2 \in F(s')$. If $\gamma_2^\star \notin \mathscr{PSF}$, then call COUNTERPATH$(G, \gamma_2, s')$, and return $s_0, ..., s_{k-1}, P', C'$ where P', C' is the result of the call; otherwise, if $\gamma_2^\star \in \mathscr{PSF}$, then complete $s_0, ..., s_k$ to any prefix-cycle path P, C having at most $|S(M)|$ states and return it.

- $\phi = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$. If there exists a prefix-cycle pair $P, C = s_0, s_1, ..., s_k$ in G such that $k < |S(M)|$ and $\gamma_2 \in F(s_i)$, for each $i = 0, ..., k$ then return P, C (this can be efficiently determined).

  In the other case, determine any state s' which is reachable from s by a path $s = s_0, ..., s_k = s'$ such that $\gamma_2 \in F(s_i)$, for all $i = 0, ..., k$ and $\gamma_1 \in F(s_k)$. Now, if both $\gamma_1^\star, \gamma_2^\star \in \mathscr{PSF}$, then complete the path $s_0, ..., s_k$ to an arbitrary prefix-cycle pair P, C such that $|P| + |S| \leqslant |S(M)|$ and return it.

  Otherwise, call COUNTERPATH(G, γ_1, s'), if $\gamma_1^\star \notin \mathscr{PSF}$, and call COUNTERPATH$(G, \gamma_2, s')$, if $\gamma_2^\star \notin \mathscr{PSF}$; note that only one of the two cases can apply. Return $P, C = s_0, ..., s_{k-1}, P', C'$ where P', C' is the result of the call.

The correctness of the procedure COUNTERPATH(G, ϕ, s) follows from the proof of Theorem 5.1. It is not hard to see that each of the cases in the body of COUNTERPATH can be completed in polynomial time (modulo recursion). Since the recursion depth is bounded by the formula length $|\phi|$, it follows that some P, C can be constructed in polynomial time. Using proper data structures (in particular for the maximal strongly conneceted components in subgraphs of R induced by labelings in F), each case can be handled in $O(|S(M)| + |R(M)|)$ time, i.e., in linear time in the size of M. Thus, the procedure COUNTERPATH(G, ϕ, s) takes $O(|\phi|(|S(M)| + |R(M)|))$ time.

Since, as remarked above, also the construction of $G=(S, R, F)$ is possible in $O(|\phi|(|S(M)|+|R(M)|))$ time, it follows that some P, C can be computed from M and ϕ in $O(|\phi|(|S(M)|+|R(M)|))$ time. This proves the second part and the result. ∎

Remarks. (1) The representation P, C of the path π returned by COUNTERPATH can be adorned to provide more information about the failure of subformulas. In particular, for an unless $\mathbf{A}(\phi_1 \mathbf{V} \phi_2)$ the stage s_k in π demonstrating the failure of $\phi_1 \mathbf{V} \phi_2$ can be marked, and similarly for an until $\mathbf{A}(\phi_1 \mathbf{U} \phi_2)$; if ϕ_2 is false in each state of π, this could be marked at $\pi(0)$. An adorned cycle-prefix pair P, C can be seen as a compact representation of a linear counterexample, which, different from a counterpath, retains all structural information of the underlying multi-path.

(2) There are instances ϕ of templates in **LIN** and structures M such that for any prefix-cycle pair P, C of an arbitrary counterpath for ϕ in M, the size $|P|+|C|$ is $\Omega(d_{\mathbf{A}}(\phi)\,|S(M)|)$; the prefix P may cycle through states in M for a number of times that is bounded by $d_{\mathbf{A}}(\phi)$, which can not be expressed by an (infinite) cycle.

We close this section with briefly addressing the problem of recognizing linear counterexamples. Even if we know that it is possible to compute some arbitrary counterpath for instances of templates in **LIN** efficiently in polynomial time, we can not infer from this that deciding whether any given counterpath is valid is possible in polynomial time. However, this problem is easily reduced to a model checking problem for arbitrary ACTL formulas, and thus solved in polynomial time.

THEOREM 5.3. *Given any formula ϕ, a structure M, and a prefix-cycle representation P, C of a path in M, deciding whether P, C is a valid counterpath for ϕ in M is possible in polynomial time (in fact, in $O(|\phi|(|P|+|C|))$ time).*

Proof. From P, C and M, we can easily construct a single-path structure M' in polynomial time by renaming states repeatedly occurring in P, C such that the i-th stages of $\pi(M')$ and P, C have the same labels for every $i \geqslant 0$. It follows that P, C is a valid counterpath iff $M' \not\models \phi$. Deciding the latter is well-known polynomial. Using the algorithm in [3], it is possible in $O(|\phi|(|S(M')|+|R(M')|))$ time. Since $|S(M')|$ and $|R(M')|$ are $O(|P|+|C|)$ and M' can be constructed in $O(|\phi|(|P|+|C|))$ time, it follows that checking validity of P, C can be done in $O(|\phi|(|P|+|C|))$ time. ∎

6. ALL *c*-LINEAR TEMPLATES ARE IN LIN

The proof of the converse of Theorem 5.1 is based on the observation that particular instantiations of non-linear templates can be used to derive the result. The structure of these instantiations allows to build structures in which no linear counterexamples exist in a systematic way.

DEFINITION 6.1 (disjoint and positive instantiation). A *disjoint instantiation* of a template $\gamma^\star \in \mathbf{T}^\star$ is an instantiation ϕ of $\gamma^\star$ which can be built starting from pure state formulas such that $\wedge$, $\vee$, $\mathbf{A}(\cdot\,\mathbf{U}\,\cdot)$, $\mathbf{A}(\cdot\,\mathbf{V}\,\cdot)$ are only applied to formulas ϕ_1 and ϕ_2 having disjoint sets of atomic propositions, i.e. $AP(\phi_1) \cap AP(\phi_2) = \varnothing$.

An instantiation ϕ is *positive*, if each occurrence of an atom in ϕ is under an even number of negations.

Notice that in a positive template instantation ϕ, each subformula $\neg\psi$ which is not in the scope of another negation is logically equivalent to a monotone (negation-free) Boolean formula over $AP(\psi)$. Observe also that $\neg\psi \not\equiv true$ and $\neg\psi \not\equiv false$ holds in this case.

Positive disjoint instantiations have the nice property that with respect to counterexamples, any part of a Boolean combination ϕ of formulas $\phi_1, ..., \phi_m$ can be "projected out" in suitable structures, i.e., to counterexamples for a simplified formula ϕ' give rise to counterexamples for ϕ. This is particularly useful for showing that ϕ is not c-linear if any of $\phi_1, ..., \phi_m$ is not c-linear.

LEMMA 6.1. *Let ϕ be a positive disjoint instantiation of $\phi^\star \in \mathbf{T}^\star$ which is a monotone Boolean combination of distinct formulas $\phi_1, ..., \phi_m$, viewed as atoms, where each ϕ_i is used only once. Let ϕ^+ be any nonempty formula obtained by omitting any atoms $\phi_1, ..., \phi_m$ in the inductive construction of ϕ. Let M^+ be any structure such that $R(M^+)$ is total and $AP(M^+) \cap AP(\phi) = AP(\phi^+)$. Then, there exists a structure M that coincides with M^+ on all components except $AP(M) = AP(M^+) \cup AP(\phi)$ and, for each state $s \in S(M)$, $L(M)(s) = L(M^+)(s) \cup P$ where $P \subseteq AP(\phi) \backslash AP(\phi^+)$, such that (1) $M, s \models \phi$ iff $M^+, s \models \phi^+$ holds for each state s, and (2) for each path π, it holds that π is a local counterpath for ϕ in M iff π is a local counterpath for ϕ^+ in M^+.*

Proof. Since ϕ is positive, all ϕ_i are positive. Thus, every formula ϕ_i which does not occur in ϕ^+ can be made either globally true in M^+, by including $AP(\phi_i)$ in the label of each state s, or globally false in M^+, by not including any atom from $AP(\phi_i)$ in the label of each state s.

Let M result from M^+ by making each ϕ_i globally true (resp., false) such that ϕ_i occurs in a maximal subformula ψ of ϕ which is omitted in the inductive construction of ϕ and connected in ϕ by conjunction (resp., disjunction), that is, ϕ has a subformula of form $\psi \wedge \psi'$ or $\psi' \wedge \psi$ (resp., $\psi \vee \psi'$ or $\psi' \vee \psi$) where all subformulas in ψ are omitted but not all subformulas in ψ'. For example, the formula $\phi = ((\mathbf{AX}(a) \vee b) \wedge \mathbf{AX}(c)) \vee (d \vee \mathbf{A}(e\mathbf{U}f))$ is a monotone Boolean combination $\phi = ((\phi_1 \vee \phi_2) \wedge \phi_3) \vee (\phi_4 \vee \phi_5)$ of "atoms" $\phi_1 = \mathbf{AX}(a)$, $\phi_2 = b$, $\phi_3 = \mathbf{AX}(c)$, $\phi_4 = d$, and $\phi_5 = \mathbf{A}(e\mathbf{U}f)$. Let $\phi^+ = \phi_3 \vee \phi_4 = \mathbf{AX}(c) \vee d$ result by omitting ϕ_1, ϕ_2, and ϕ_5 in the construction of ϕ. Then, given a structure M^+ with total $R(M^+)$ such that $AP(M^+) \cap AP(\phi) = \{c, d\}$, we obtain M by adding a and b to the label of each state in M^+ (this effects that ϕ_1 and ϕ_2 are globally true in M, while ϕ_5 is globally false).

It is not hard to see that the structure M so constructed satisfies the property stated in the lemma. ▮

The next lemma informally states that for any positive disjoint instantiation of a template in **LIN**, we can always find a structure that permits only one path and such that the formula is true in it, but false if we proceed long enough along this path. For example, consider the instantiation $\gamma = \mathbf{A}(a\mathbf{U}b)$ of the template $\mathbf{A}(\star\mathbf{U}\star)$ and the structure M corresponding to the transition graph in Fig. 6.

Clearly $M \models \gamma$, since γ is true along the unique path $\pi = [s_0, s_1, s_1, ...]$ in M. However, it is sufficient to proceed just one stage along π to make γ false; in fact, γ fails in each suffix π^i for $i \geqslant 1$.

Observe that the above property does not hold for all instantiations of templates in **LIN**. For example, consider the instance $\phi = \mathbf{A}(\mathit{false}\mathbf{V}a)$ of the template $\mathbf{A}(\bigstar\mathbf{V}\bigstar)$, which belongs to **LIN**. A counterexample for ϕ is a path π along which a is false in some state $\pi(i)$. Here, it is impossible to prefix π with a sequence $s_0, ..., s_k$ of states such that along the resulting path $\mathit{false}\mathbf{V}a$ becomes true.

Before we state the lemma, we need some preliminary definition. Recall that a structure is conic, if it has a single initial state and this state is not reachable from any state of the structure (see Section 2).

Definition 6.2 (single-path structure). A conic structure M is a *single-path* structure, if M has a single path starting at the initial state, and each state in M occurs in it. We denote this path by $\pi(M)$.

An immediate consequence of this definition is that for any single-path structure M and non pure-state formula γ it holds that $M \not\models \gamma$ just in case where $\pi(M)$ is a counterpath for γ.

Lemma 6.2. *For every positive disjoint instantiation γ of a template $\gamma^\star \in$ **LIN**, there exists a single-path structure M and a $k \geqslant 1$ such that $M \models \gamma$ and $\pi(M)^k$ is a local counterpath for γ (resp., $\pi(M)(k) \not\models \gamma$ if $\gamma^\star \in \mathscr{PSF}$).*

Proof. By induction on the stage $i \geqslant 0$ of $\Lambda^i = (S_1^i, S_2^i)$ in which $\gamma^\star$ first occurs (see Appendix A). ∎

The next lemma informally says that for any positive disjoint instantiation γ of a template in **LIN**, it is possible to find a single-path structure which does not satisfy γ, but γ is always satisfied if we proceed long enough on the single path. This lemma is in a sense complementary to the previous lemma. Similar as there, the property is not true for arbitrary instantiations of templates from **LIN**. E.g., a single-path structure falsifying $\gamma = \mathbf{A}(\mathit{true}\mathbf{U}a)$ does not contain any "suffix" structure in which γ holds.

Prior to the lemma, we introduce the notion of k-structure.

Definition 6.3 (k-structure). A *k-structure* for a positive disjoint instantiation γ of a template $\gamma^\star \in \mathbf{T}^\star$ is any conic structure M such that $M \not\models \gamma$ and for each path π in M starting at s_0, there exists an index $k \geqslant 1$ such that $M, \pi^i(0) \models \gamma$, for each $i \geqslant k$.

We will use k-structures repeatedly in constructions of structures which do not have linear counterexamples for formulas involving the until operator.

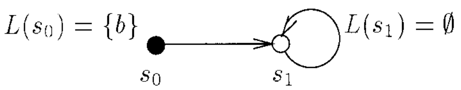

FIG. 6. Transition graph representing structure M (initial state s_0).

Lemma 6.3. *Each positive disjoint instantiation γ of any template $\gamma^\star \in \mathbf{LIN}$ has some single-path k-structure M.*

Proof. By induction on the stage $i \geqslant 0$ of $\Lambda^i = (S_1^i, S_2^i)$ in which $\gamma^\star$ first occurs (see Appendix A). ∎

In the next result, we show that a large class of templates in $\mathbf{T}^\star \backslash \mathbf{LIN}$ which involve nesting into the until operator $\mathbf{U}$ or the unless operator $\mathbf{V}$, respectively, are not linear. We establish this by proving that positive disjoint instantiations of these templates are not c-linear. We introduce some preliminary concepts.

Definition 6.4 (left- and right-structures). A *left-structure M* for a positive disjoint instantiation $\phi = \mathbf{A}\psi$ of a template in $\mathbf{T}^\star$ is a conic structure with initial state s_0 and $AP(M) = AP(\phi)$, which satisfies, depending on the linear-time operator guarding ψ, the following properties (see Fig. 7):

If $\phi = \mathbf{AX}(\phi_1)$, then only one transition (s_0, s'_0) leaving from s_0 exists, and

- s'_0 is the initial state of a structure Π_{ϕ_1}, contained in M, such that $\Pi_{\phi_1} \not\models \phi_1$,
- s_0 does not appear in the set of states of Π_{ϕ_1}.

If $\phi = \mathbf{A}(\phi_1 \mathbf{V} \phi_2)$, then

- s_0 is the initial state of a structure Π_{ϕ_1}, contained in M, such that $\Pi_{\phi_1} \not\models \phi_1$;
- there is only one transition from s_0 to a state s'_0 not belonging to Π_{ϕ_1}, which is the initial state of a structure Π_ϕ, contained in M, such that $\Pi_\phi \not\models \phi$;
- the sets of states of Π_{ϕ_1} and Π_ϕ are disjoint.

For $\phi = \mathbf{A}(\phi_1 \mathbf{U} \phi_2)$, M is similar as for $\phi = \mathbf{A}(\phi_1 \mathbf{V} \phi_2)$, but with the roles of ϕ_1 and ϕ_2 exchanged. *Right-structures* for ϕ are particular left-structures, such that all structures Π_ϕ, Π_{ϕ_1}, and Π_{ϕ_2} involved—with the exception of Π_{ϕ_1} for $\phi = \mathbf{A}(\phi_1 \mathbf{V} \phi_2)$—are k-structures (see Fig. 8).

Left- and right-structures will be used as components for the left-nested and right-nested formulas γ_1 and γ_2, respectively, in the constructions of structures M witnessing the fact that formulas $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$ are not c-linear in general.

Lemma 6.4. *Let $\phi = \mathbf{A}\psi$ be a positive disjoint instantiation of some template $\gamma^\star \in \mathbf{T}^\star$. Then, there exists some right-structure for ϕ, and if $\gamma^\star \in \mathbf{LIN}$, there exists also some left-structure for ϕ.*

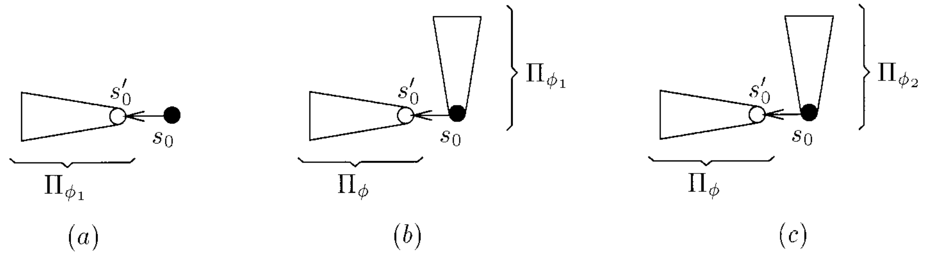

FIG. 7. Left-structures for (a) $\phi = \mathbf{AX}\phi_1$, (b) $\phi = \mathbf{A}(\phi_1 \mathbf{V} \phi_2)$, and (c) $\phi = \mathbf{A}(\phi_1 \mathbf{U} \phi_2)$.

Proof. Left-structures for ϕ are easily constructed (use, e.g., the technique of making formulas globally false in Lemma 6.1 to construct the substructures Π_ϕ, Π_{ϕ_1}, and Π_{ϕ_2} of M). If $\gamma^\star \in \mathbf{LIN}$, then the subtemplates $\gamma_1^\star$, $\gamma_2^\star$ of $\gamma^\star = \mathbf{AX}\gamma_1^\star$ resp. $\gamma^\star = \mathbf{A}(\gamma_1^\star \mathbf{U} \gamma_2^\star)$, $\gamma^\star = \mathbf{A}(\gamma_1^\star \mathbf{V} \gamma_2^\star)$ belong to $\mathbf{LIN}$ as well. By Lemma 6.3, we can thus use single-path k-structures for the substructures, and thus the resulting left-structure is also a right-structure. ∎

We note the following proposition.

PROPOSITION 6.5. *Let M be any left-structure for a positive disjoint instantiation $\phi = \mathbf{A}\psi$ of a template in $\mathbf{T}^\star$. Then $M \not\models \phi$.*

Proof. For $\phi = \mathbf{AX}(\phi_1)$, this is obvious. To see this for $\phi = \mathbf{A}(\phi_1 \mathbf{V} \phi_2)$, let Π be a counterexample for ϕ in Π_ϕ (which exists by Theorem 3.1), and let Π_0 be a counterexample for ϕ_1 in Π_{ϕ_1} (starting at s_0). Then, the multi-path $[\Pi_0, \Pi(0), \Pi(1), ...]$ is a counterexample for ϕ. In case $\phi = \mathbf{A}(\phi_1 \mathbf{U} \phi_2)$, let Π be a counterexample for ϕ in Π_ϕ (which exists by Theorem 3.1), and let Π_0 be a counterexample for ϕ_2 in Π_{ϕ_2} (starting at s_0). Then, the multi-path $[\Pi_0, \Pi(0), \Pi(1), ...]$ is a counterexample for ϕ. ∎

The following definition introduces a formal notion of merging two conic structures at their initial states, which is used repeatedly in the proofs of the subsequent results of this section.

DEFINITION 6.5 (fusion of structures). Let M_1 and M_2 be conic structures with initial states s_0^1 and s_0^2, respectively, having disjoint sets of states. Then, the *fusion* of M_1 and M_2 is the conic structure M obtained by taking the union of M_1 and M_2, where s_0^1 and s_0^2 are merged into a single state s_0 with label $L(s_0) = L(s_0^1) \cup L(s_0^2)$.

THEOREM 6.6. *Let γ be a positive disjoint instantiation of a template $\gamma^\star \in \mathbf{T}^\star$ such that either*

1. $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, *where* $\gamma_1^\star \notin \mathcal{PSF}$ *and* $\gamma_2^\star \in \mathbf{LIN} \backslash \mathcal{PSF}$, *or*
2. $\gamma = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$, *where* $\gamma_1^\star \notin \mathcal{PSF}$, *and* $\gamma_2^\star \in \mathbf{LIN}$.

Then, γ is not c-linear.

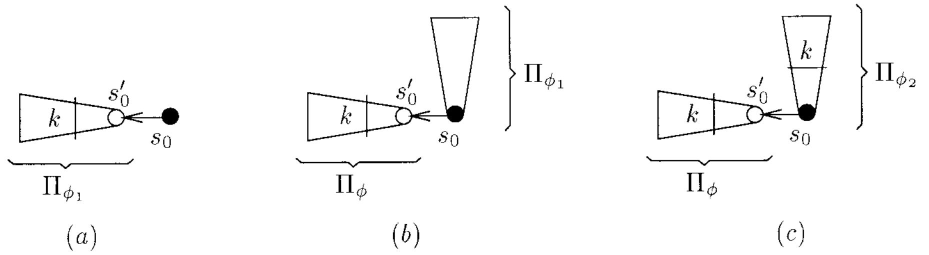

FIG. 8. Right-structures for (a) $\phi = \mathbf{AX}\phi_1$, (b) $\phi = \mathbf{A}(\phi_1 \mathbf{V} \phi_2)$, and (c) $\phi = \mathbf{A}(\phi_1 \mathbf{U} \phi_2)$.

Proof. We have to find a structure M such that both $M \not\models \gamma$ and each counterexample for γ in M is not a linear multi-path. We prove the statement first for the case in which γ_1 and γ_2 are of the form $\mathbf{A}\psi$ or, for item 2, γ_2 is a pure state formula. By exploiting Lemma 6.1, we can then conclude that the statement is true in general.

We will construct M for item 1 starting from a left-structure M_1 and a right-structure M_2 for the subformula γ_1 and γ_2, respectively. Observe that, by Lemma 6.4, such M_1 and M_2 exist; unless stated otherwise, we assume that they have disjoint sets of states. For item 2, we will construct M starting from a single-path structure for γ as in Lemma 6.2.

(1) $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1^\star = \mathbf{A}\psi_1^\star \notin \mathscr{PSF}$ and $\gamma_2^\star = \mathbf{A}\psi_2^\star \in \mathbf{LIN} \backslash \mathscr{PSF}$. We construct M as the fusion of a left-structure M_1 for γ_1 and a right-structure M_2 for γ_2 with initial state s_0, and modify M according to the linear time operators $\mathbf{X}$, $\mathbf{V}$, and $\mathbf{U}$, guarding ψ_1 and ψ_2, respectively. Out of the nine emerging cases, we consider here two cases; the others are similar (see Appendix A).

- $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$ and $\gamma_2 = \mathbf{AX}(\gamma_{2,1})$. We modify M as follows. In each state s of the structure $\Pi_{\gamma_{1,1}}$ in M_1 (see Def. 6.4), we include $AP(\gamma_{2,1})$ (i.e., in its label $L(s)$), and in each state of $\Pi_{\gamma_{2,1}}$ in M_2, we include $AP(\gamma_{1,1})$ (see Fig. 9).

Clearly, these additions preserve the existence of counterexamples for $\gamma_{1,1}$ in $\Pi_{\gamma_{1,1}}$ and for $\gamma_{2,1}$ in $\Pi_{\gamma_{2,1}}$, respectively, since $AP(\gamma_{1,1})$ and $AP(\gamma_{2,1})$ are disjoint.

It holds that $M \not\models \gamma$, since $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$. Indeed, we can find a counterexample for $\gamma_1 \vee \gamma_2$ simply by merging a counterexample for γ_1 in M_1 with a counterexample for γ_2 in M_2. Clearly, this counterexample is not linear.

It remains to show that no linear counterexample for γ in M exists. First observe that no counterexample for γ_1 is in M_2. Indeed, for every multi-path Π in M_2, $\Pi(1)$ cannot be a counterexample for $\gamma_{1,1}$, since each state of M_1 except s_0 contains the set $AP(\gamma_{1,1})$. Similarly, there is no counterexample for γ_2 is in M_1. Hence, each counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. By Definition 3.5, any counterexample for γ must involve counterexamples for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. Clearly, this concludes the proof. Towards a contradiction, suppose Π is a linear counterexample such that $\Pi(i)$ is a counterexample for γ_2, for every $i \geqslant 0$. Since γ_2 is globally true in M_1, $\mu(\Pi)$ must lead into M_2, and thus into $\Pi_{\gamma_{2,1}}$. However, $\Pi_{\gamma_{2,1}}$ is a k-structure, which means that γ_2 is eventually true. This raises the desired contradiction.

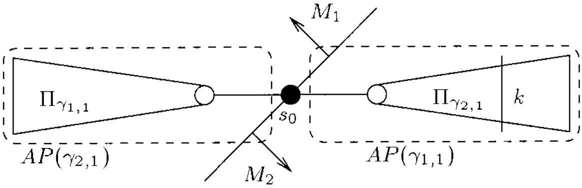

FIG. 9. The **X**-**X** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$ and $\gamma_2 = \mathbf{AX}(\gamma_{2,1})$.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1}\mathbf{V}\gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1}\mathbf{U}\gamma_{2,2})$. We modify M as follows. We add (1) to every state of M_1 except s_0 the set $AP(\gamma_2)$; (2) to every state of M_2 except s_0 the set $AP(\gamma_1)$; (3) to s_0 the set $AP(\gamma_{1,2}) \cup AP(\gamma_{2,1})$; and, (4) in every state $s \neq s_0$ of $\Pi_{\gamma_{2,2}}$ in M_2 the set $AP(\gamma_{2,1})$ (see Fig. 10; note that the order of these additions is immaterial).

It easy to see that, also after these additions, $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_1 is in M_2. Indeed, a counterexample for γ_1 must contain a counterexample for $\gamma_{1,2}$. But this is impossible, since each state in M_2 contains $AP(\gamma_{1,2})$. Finally, no counterexample for γ_2 is in M_1. Indeed, since each state of M_1 contains the set $AP(\gamma_{2,1})$, a counterexample for γ_2 in M_1 could only be a multi-path Π such that each element $\Pi(i)$ is a counterexample for $\gamma_{2,2}$, for each $i \geqslant 0$. But this is impossible, since $\gamma_{2,2}$ is globally true in M_1.

Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. By Definition 3.5 a counterexample for γ must involve a counterexample for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. This, clearly, concludes the proof. Towards a contradiction, suppose Π is such a linear counterexample, i.e., $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$. But such a counterexample cannot be linear. Indeed, it cannot lead into Π_{γ_2}, since this is a k-structure of γ_2. On the other hand, it cannot lead into $\Pi_{\gamma_{2,2}}$. Indeed, a counterexample for γ_2 cannot involve a counterexample for $\gamma_{2,1}$ as $\Pi_{\gamma_{2,2}}$ contains in each state the set $AP(\gamma_{2,1})$. Thus, such a counterexample could only be a multi-path Π such that $\Pi(i)$ is a (linear) counterexample for $\gamma_{2,2}$, for each $i \geqslant 0$. But this is not possible, since $\Pi_{\gamma_{2,2}}$ is a k-structure of $\gamma_{2,2}$. Hence, no counterexample for γ in M is linear.

(2) The second case is $\gamma = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$, where $\gamma_1^\star = \mathbf{A}\psi_1^\star \notin \mathscr{PSF}$ and either $\gamma_2^\star = \mathbf{A}\psi_2^\star \in \mathbf{LIN}$ or $\gamma_2^\star \in \mathscr{PSF}$.

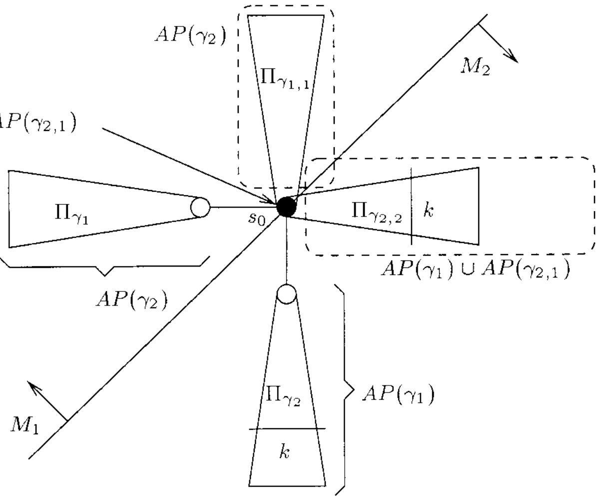

FIG. 10. The **V-U** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{A}(\gamma_{1,1}\mathbf{V}\gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1}\mathbf{U}\gamma_{2,2})$.

For each possible shape of the template $\gamma_1^\star$, we construct a structure M such that both $M \not\models \gamma$ and each counterexample for γ in M is not linear. The structure M is obtained by a modification of the structure M_γ which we define next.

Let M' be a single-path structure as stated in Lemma 6.2 for formula γ_2. Thus, $M' \models \gamma_2$. Furthermore, there exists an index $k \geqslant 1$ such that $\pi(M')^k$ is a local counterpath for γ_2 (resp., γ_2 is false in $\pi(M')(k)$). Without loss of generality, k is the least index having this property. Denote by $s_i = \pi(M')(i)$, for $i = 0, ..., k$, the first $k+1$ states appearing in $\pi(M')$. Note that the s_i (hence also the suffixes $\pi(M')^i$) are pairwise distinct. Furthermore, s_k is the initial state of a structure M^+ induced by s_k in M' (i.e., the suffix $\pi(M')^k$) such that $M^+ \not\models \gamma_2$.

Let M_0 be a left-structure for γ_1. We take copies $M_1, ..., M_{k-1}$ and repeatedly take the fusion of M_i with the substructure of M' induced by the state s_i in M', for $i = 0, ..., k-1$. The repeatedly so revised structure M' is the desired structure M_γ with initial state s_0 (cf. Fig. 11).

We consider here one of the three emerging types of $\gamma_1^\star$; the proof in the other cases is similar (see Appendix A).

- $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$, i.e., $\gamma = \mathbf{A}(\mathbf{AX}(\gamma_{1,1})\mathbf{V}\gamma_2)$. To construct M, we modify the above structure M_γ as follows. Include in the label of each state not appearing in $\pi(M')^k$ the set $AP(\gamma_2)$. Note that this addition does not affect the existence of (local) counterexamples for γ_1 starting at $s_0, s_1, ..., s_{k-1}$, since $AP(\gamma_1)$ and $AP(\gamma_2)$ are disjoint. Finally, we add the set $AP(\gamma_1)$ in every state of M' (thus, to each state appearing in $\pi(M')$). This addition preserves the existence of counterexamples for γ_1 starting with $s_0, s_1, ..., s_{k-1}$, since γ_1 involves the next-time operator. Furthermore, $\pi(M')^k$ is still a local counterpath for γ_2, since $AP(\gamma_1)$ and $AP(\gamma_2)$ are disjoint. The resulting conic structure with initial state s_0 is M (see Fig. 11).

We can see that $M \not\models \gamma$. Indeed, there exists a multi-path Π_2, such that $\Pi_2(i)$ is a ℓ-counterexample for γ_1, for $0 \leqslant i \leqslant k-1$ (recall that each state s_i is origin of a ℓ-counterexample for γ_1), and $\Pi_2(k)$ is a local counterexample for γ_2 with main path $\pi(M')^k$. Clearly, this multi-path is not linear. Moreover, no linear counterexample for γ is in M. Indeed, each counterexample for γ needs a counterexample

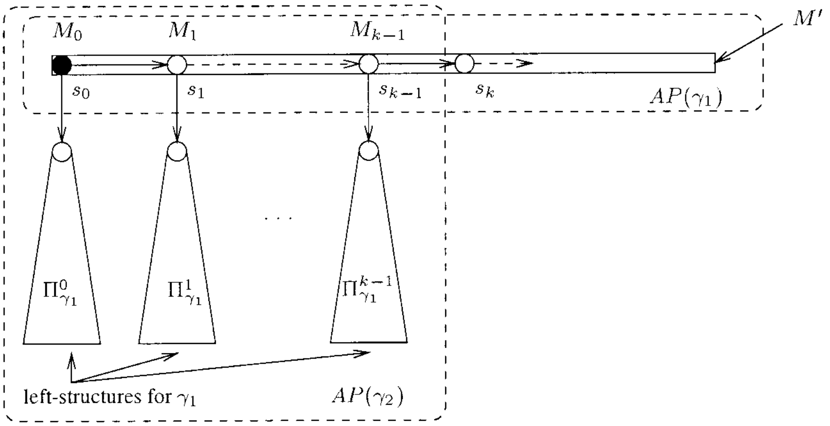

FIG. 11. Nesting into unless, the $\mathbf{X}$ case: $\gamma = \mathbf{A}(\mathbf{AX}(\gamma_{1,1})\mathbf{V}\gamma_2)$.

for γ_2. But all paths starting with the initial state s_0 cannot be a local counterpath for γ_2. Indeed, each path π not reaching states beyond s_k cannot be a counterpath for γ_2, since the label of each state appearing in π would contain the set $AP(\gamma_2)$. On the other hand, the only path starting with s_0 and reaching s_k is $\pi(M')$. However, as M' was chosen according to Lemma 6.2, this path cannot be a counterpath for γ_2. Hence, we need a counterexample whose first element is a counterexample for γ_1. Clearly, we cannot find a counterexample for γ_1 along the path $\pi(M')$, since each state in it contains $AP(\gamma_1)$. Hence, each counterexample for γ necessarily contains branching, that is, it is not linear.

This concludes the proof for the case in which $\gamma_1^\star, \gamma_2^\star$ have form $\mathbf{A}\psi$ or $\gamma_2^\star \in \mathcal{PSF}$. For the case of a general $\gamma_1^\star$, Lemma 6.1 can be exploited: the instantiation γ_1 is a monotone Boolean combination of positive disjoint instantiations $\gamma_{1,1}, ..., \gamma_{1,m}$ (each of which occurs only once) such that w.l.o.g. $\gamma_{1,1}$ is of the form $\mathbf{A}\psi$. We proceed then for γ_1 as for $\gamma_{1,1}$, but use the structure M from Lemma 6.1 for $\phi = \gamma_1$ instead of the structure M^+ for $\phi^+ = \gamma_{1,1}$ (observe that M^+ can always be chosen such that $R(M^+)$ is total). For the general case of γ_2, we proceed analogously. This proves the result. ▮

THEOREM 6.7. *Let γ be any positive disjoint instantiation of a template $\gamma^\star \in \mathbf{T}^\star$. If $\gamma^\star \notin \mathbf{LIN}$, then γ is not c-linear.*

Proof. We proceed by induction on the number of universal quantifiers $\mathbf{A}$ appearing in γ, which is denoted by $n_{\mathbf{A}}(\gamma)$.

(Basis) The case $n_{\mathbf{A}}(\gamma) = 0$ is trivial, since $\gamma^\star$ belongs to $\mathcal{PSF} \subseteq \mathbf{LIN}$.

(Induction) Assume that the statement is true for every γ such that $n_{\mathbf{A}}(\gamma) < k$. We have to show that each positive disjoint instantiation γ of $\gamma^\star \in \mathbf{T}^\star \backslash \mathbf{LIN}$ such that $n_A(\gamma) = k$ is not c-linear, i.e., that there is a structure M such that both $M \not\models \gamma$ and each counterexample for γ in M is not linear.

The formula γ is either of the form $\mathbf{A}\psi$, or a Boolean combination of formulas $\gamma_1, ..., \gamma_m$. We consider the possible cases.

- $\gamma = \mathbf{AX}\phi$, where $n_{\mathbf{A}}(\phi) = k-1$. By Definition 4.9, $\gamma^\star \notin \mathbf{LIN}$ if and only if $\phi^\star \notin \mathbf{LIN}$. Thus, since $n_{\mathbf{A}}(\phi) = k-1$, the induction hypothesis implies that ϕ is not c-linear. Hence, there exists a structure M' such that both $M' \not\models \phi$ and no counterexample for ϕ in M' is linear. Without loss of generality, M' is conic and has the initial state s_0'. Let the conic structure M with initial state s_0 result by connecting a new state s_0 to M' via the transition (s_0, s_0'). Clearly, $M \not\models \gamma$. Furthermore, each counterexample Π for γ is such that $\Pi(1)$ is a counterexample for ϕ. Since $or(\Pi(1)) = s_0'$, $\Pi(1)$ cannot be linear, by hypothesis. Hence, Π is not linear.
- $\gamma = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$, where $n_{\mathbf{A}}(\gamma_1) + n_{\mathbf{A}}(\gamma_2) = k-1$. By the definition of $\mathbf{LIN}$, the following two cases cover each γ such that $\gamma^\star \notin \mathbf{LIN}$:

1. $\gamma_1^\star \notin \mathcal{PSF}$ and $\gamma_2^\star \in \mathbf{LIN}$. This case has been already proven in Theorem 6.6.

2. $\gamma_2^\star \notin \mathbf{LIN}$. By the induction hypothesis, γ_2 is not c-linear. Thus, there exists a structure M such that $M \not\models \gamma_2$ and no counterexample in M is linear. We

modify M by adding in each state the set $AP(\gamma_1)$. Clearly, no local counterexamples for γ_1 can be found in M. However, $M \not\models \gamma$. Moreover, each counterexample for γ in M must start with a counterexample for γ_2. Hence, it is not linear.

- $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $n_{\mathbf{A}}(\gamma_1) + n_{\mathbf{A}}(\gamma_2) = k-1$. Due to the intricate possibilities of nesting into an until from **LIN**, this case requires a careful analysis of several subcases. The following cases exhaust each possibility of $\gamma^\star \notin \mathbf{LIN}$:

1. $\gamma_1^\star \notin \mathcal{PSF}$ and $\gamma_2^\star \in \mathbf{LIN} \backslash \mathcal{PSF}$;
2. $\gamma_1^\star \notin \mathbf{LIN}$ and $\gamma_2^\star \in \mathcal{PSF}$;
3. $\gamma_2^\star \notin \mathbf{LIN}$;
4. $\gamma_1^\star \in \mathcal{PSF}$ and $\gamma_2^\star \in \mathbf{LIN} \backslash (S_2^\infty \cup \mathcal{PSF})$.

Case 1 is already proved by Theorem 6.6, and cases 2,3 are simple to prove from the induction hypothesis. For the remaining case 4, we conclude from Lemma 6.1 that it is sufficient to consider the following cases for $\gamma_2^\star$, where $\mathbf{ULIN} = \mathbf{LIN} \cap \mathbf{AU}(\mathbf{T}^\star, \mathbf{T}^\star)$ is the set of all linear until templates:

4.1. $\gamma_2^\star \in \mathbf{AX}(\mathbf{LIN})$.

4.2. $\gamma_2^\star \in \mathbf{AV}(\mathcal{PSF}, \mathbf{LIN})$;

4.3. $\gamma_2^\star \in \mathbf{ULIN} \wedge \mathbf{ULIN}$;

4.4. $\gamma_2^\star \in (\mathcal{PSF} \wedge \mathbf{ULIN}) \cup (\mathbf{ULIN} \wedge \mathcal{PSF})$.

Indeed, γ_2 is a positive disjoint instantiation of $\gamma_2^\star$ that can be viewed as monotone Boolean combination of different atoms $\phi_1, \ldots, \phi_n$ where each ϕ_i is either a pure state formula or of form $\mathbf{A}\psi$. Since $\gamma_2^\star \notin \mathcal{PSF}$, for some ϕ_i either (i) $\phi_i = \mathbf{AX}\psi_1$, (ii) $\phi_i = \mathbf{A}(\psi_1 \mathbf{V} \psi_2)$, or (iii) $\phi_i = \mathbf{A}(\psi_1 \mathbf{U} \psi_2)$. If in case (iii) neither $\mathbf{X}$ nor $\mathbf{V}$ occur in γ_2, then some ϕ_j where $i \neq j$ must exist such that $\phi_j^\star \in \mathcal{PSF} \cup \mathbf{ULIN}$ and the common ancestor of ϕ_i and ϕ_j in the formula tree of γ_2 is a conjunction node (i.e., ϕ_i and ϕ_j are subformulas of formulas α and β, respectively, such that $\alpha \wedge \beta$ is a subformula of γ_2). Lemma 6.1 implies that it is sufficient to consider the formula ϕ^+ where $\phi^+ = \phi_i$ in cases (i), (ii) and $\phi^+ = \phi_i \wedge \phi_j$ in case (iii). Indeed, no disjunction $\phi^+ = \phi_i \vee \phi_j$ needs to be considered: if no ϕ^+ as in the cases (i)–(iii) exists, then $\phi_j^\star \in \mathbf{ULIN}$ must hold for at least one ϕ_j where $j \neq i$. (Otherwise, all ϕ_j where $j \neq i$ would be pure state formulas and connected in γ_2 with disjunction. Thus, $\gamma_2^\star \in S_2^\infty$ and $\gamma^\star \in \mathbf{LIN}$ would hold). However, disjunction of any two templates containing subtemplates from **ULIN** clearly yields a template outside **LIN**, which would imply $\gamma_2^\star \notin \mathbf{LIN}$. Now by Lemma 6.1, for any stucture M^+ for ϕ^+ such that $R(M^+)$ is total and $AP(M^+) \cap AP(\gamma_2) = AP(\phi^+)$, we can build a structure M for $\phi = \gamma_2$ such that local counterpaths for ϕ^+ in M^+ coincide with local counterpaths for γ_2 in M. In particular, we obtain that if M^+ has no counterpath for $\mathbf{A}(\gamma_1 \mathbf{U} \phi^+)$, then M has no counterpath for $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$.

We now describe structures for 4.1–4.4 proving the claim (see Appendix A for details).

4.1. $\gamma_2^\star \in \mathbf{AX}(\mathbf{LIN})$. Let M' be a single-path structure and $k \geqslant 1$ for formula γ_2 as described in Lemma 6.2. Thus, $M' \models \gamma_2$, and $\pi(M')^k$ is a local counterpath for γ_2 (resp., $M', \pi(M')(k) \not\models \gamma_2$). Let k w.l.o.g. be the least such index.

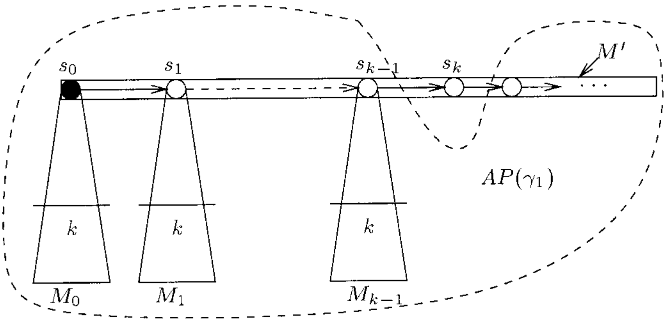

FIG. 12. Nesting of $\mathscr{PSF}$ and $\mathbf{AX}(\mathbf{T}^\star)$ into until: $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \mathbf{AX}(\gamma_{2,1}))$.

Let $s_0, s_1, ..., s_k$ denote the first $k+1$ states in $\pi(M')$. These s_i are pairwise distinct. Clearly, s_k is the first state of the suffix $\pi(M')^k$. We assume w.l.o.g. $L(M')(s_k) \cap AP(\gamma_1) = \varnothing$. Let M_0 be a k-structure for γ_2 such that the initial state has an empty label. Lemma 6.3 implies that such an M_0 exists; observe that $M_0 \not\models \gamma_2$. Let $M_1, ..., M_{k-1}$ be copies of M_0. For $i = 0, ..., k-1$ we repeatedly take the fusion of M_i with the structure induced by s_i in M'. Finally, we add to every state except s_k the set $AP(\gamma_1)$. The resulting structure is the desired M (see Fig. 12).

4.2. $\gamma_2^\star \in \mathbf{AV}(\mathscr{PSF}, \mathbf{LIN})$. Thus, $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$, where $\gamma_{2,1}$ is a pure state formula and $\gamma_{2,2}$ is c-linear by Theorem 5.1.

Let M be a k-structure for γ_2 with initial state s'_0. Such a structure exists by Lemma 6.3, and w.l.o.g. $AP(M) \cap AP(\gamma_1) = \varnothing$. We modify M by adding a new initial state s_0 with empty label and the transitions (s_0, s'_0) and (s_0, s_0). Then, we add to each state the set $AP(\gamma_1)$ and to s_0 the set $AP(\gamma_{2,2})$ (see Fig. 13).

4.3. $\gamma_2^\star \in \mathbf{ULIN} \wedge \mathbf{ULIN}$. Thus, $\gamma_2^\star = \phi_1 \wedge \phi_2$, where $\phi_1 = \mathbf{A}(\phi_{1,1} \mathbf{U} \phi_{1,2})$ and $\phi_2 = \mathbf{A}(\phi_{2,1} \mathbf{U} \phi_{2,2})$; moreover, each $\phi_{i,j}$, $i, j \in \{1, 2\}$ is an instantiation of a template in **LIN**.

We construct M as follows. Let M' be a single-path structure as in Lemma 6.2 for formula ϕ_2. Thus, $M' \models \phi_2$. Furthermore, there exists a $k \geqslant 1$ such that $\pi(M')^k$ is a local counterpath for ϕ_2 (and hence for γ_2). Let k w.l.o.g. be the least such index. Let $s_0, s_1, ..., s_k$, denote the first $k+1$ states appearing in $\pi(M')$; observe that they are pairwise distinct. Clearly, s_k is the first state of the suffix $\pi(M')^k$. Since γ is a positive disjoint instantiation, we may assume that no atom from $AP(\gamma_1) \cup AP(\phi_1)$ occurs in any state of M'.

Let M_0 be a right-structure for ϕ_1; since $\phi_1^\star \in \mathbf{LIN}$, such a structure exists by Lemma 6.4. We remark that, by definition of right-structure, $M_0 \not\models \phi_1$. Let $M_1, ..., M_{k-1}$ be copies of M_0. For $i = 0, ..., k-1$ we repeatedly take the fusion of M_i and the structure induced by the state s_i in M'. Next, we add in every state

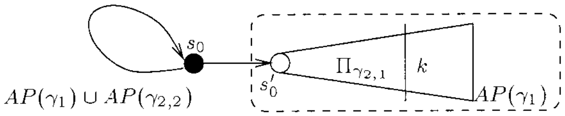

FIG. 13. Nesting of $\mathscr{PSF}$ and $\mathbf{AV}(\mathbf{T}^\star)$ into until: $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2}))$.

$s_0, ..., s_{k-1}$ the set $AP(\phi_{1,1})$. Note that after this addition, each structure M_i still satisfies $M_i \not\models \phi_1$, for $i = 1, ..., k-1$. Indeed, since $AP(\phi_{1,1}) \cap AP(\phi_{1,2}) = \emptyset$ for $\Pi^i_{\phi_{1,2}}$, still $\Pi^i_{\phi_{1,2}} \not\models \phi_{1,2}$ holds.

Now we add in every state of M_i, for $0 \leqslant i \leqslant k-1$, including states $s_0, ..., s_{k-1}$, the set $AP(\gamma_1)$. Since $AP(\gamma_1) \cap AP(\gamma_2) = \emptyset$, this has no effect on the properties of M_i from above. Moreover, we add in every state of M_i, for $0 \leqslant i \leqslant k-1$, except the states $s_0, ..., s_{k-1}$, the set $AP(\phi_2)$. Since $AP(\phi_1) \cap AP(\phi_2) = \emptyset$, this addition preserves the existence of counterexamples for ϕ_1 in the M_i's. Finally, we add $AP(\phi_1)$ in every state occurring in the path $\pi(M')^k$. After this addition, $\pi(M')^k$ is still a local counterpath for ϕ_2. The resulting structure is the desired M (see Fig. 14).

4.4. $\gamma_2^\star \in (\mathscr{PSF} \wedge \mathbf{ULIN}) \cup (\mathbf{ULIN} \wedge \mathscr{PSF})$. Thus, $\gamma_2 = \phi_1 \wedge \phi_2$. Assume that ϕ_1 is a pure state formula and $\phi_2 = \mathbf{A}(\phi_{2,1} \mathbf{U} \phi_{2,2})$, where $\phi_{2,1}$ and $\phi_{2,2}$ are instantiations of templates in **LIN**. The other case (vice versa) is similar.

Let M' be a right-structure for $\phi_2 = \mathbf{A}(\phi_{2,1} \mathbf{U} \phi_{2,2})$. We modify M' by adding $AP(\gamma_1) \cup AP(\phi_1)$ to each state and by further adding $AP(\phi_{2,1})$ to s_0; after this modification $M' \not\models \phi_2$ still holds. We now add and label two new states s_1, s_2 to obtain the desired M as shown in Fig. 15.

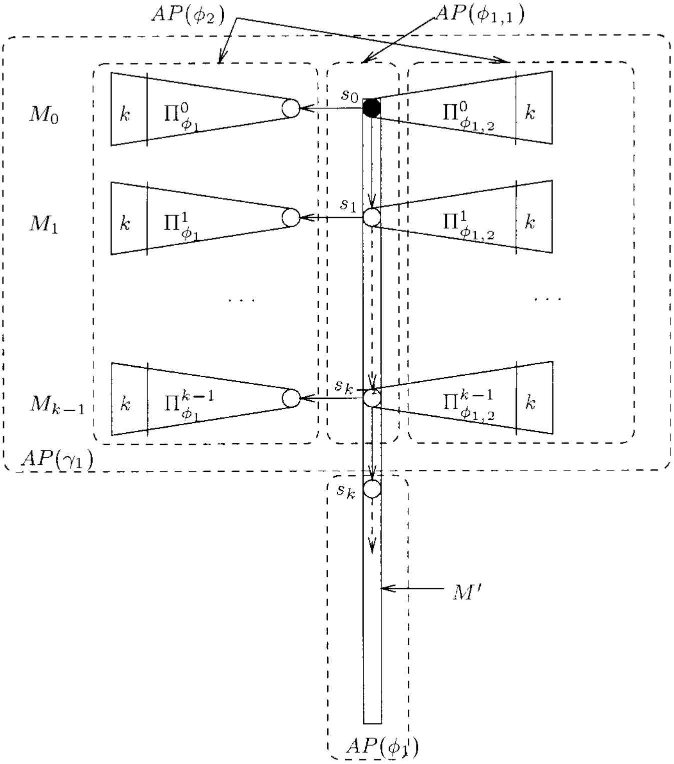

FIG. 14. Right-nesting of **ULIN** $\wedge$ **ULIN** into until: $\gamma = \mathbf{A}(\gamma_1 \mathbf{U}(\phi_1 \wedge \phi_2))$, where $\phi_1 = \mathbf{A}(\phi_{1,1} \mathbf{U} \phi_{1,2})$ and $\phi_2 = \mathbf{A}(\phi_{2,1} \mathbf{U} \phi_{2,2})$.

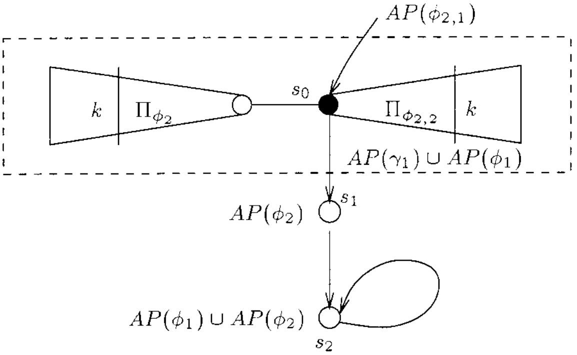

FIG. 15. Right-nesting of $\mathscr{PSF} \wedge$ **ULIN** into until: $\gamma = \mathbf{A}(\gamma_1 \mathbf{U}(\phi_1 \wedge \phi_2))$, $\phi_2 = \mathbf{A}(\phi_{2,1}\mathbf{U}\phi_{2,2})$.

- $\gamma^\star = \phi_1^\star \wedge \phi_2^\star$ or $\gamma^\star = \phi_1^\star \vee \phi_2^\star$, where $n_{\mathbf{A}}(\phi_1^\star) + n_{\mathbf{A}}(\phi_2^\star) = k$. Thus, γ can be viewed as a monotone Boolean combination of formulas $\gamma_1, ..., \gamma_m$. By applying Lemma 6.1, if one of the γ_i is not c-linear either by the induction hypothesis or by one of the already considered cases, then γ is not c-linear as well. To complete the proof, by the inductive definition of **LIN** and Lemma 6.1 it remains to consider the case $\gamma = \gamma_1 \vee \gamma_2$ where $\gamma_1^\star = \mathbf{A}\psi_1^\star \in$ **LIN** and $\gamma_2^\star = \mathbf{A}\psi_2^\star \in$ **LIN**.

We construct a conic structure M whose labeling depends on the outermost linear-time operators in $\gamma_1^\star$ and $\gamma_2^\star$. Commutativity of logical conjunction implies that six cases of conjunctions involving **AX**, **AU**, and **AV** remain to be considered. We do this for **AU** and **AU**; the other cases are similar (see Appendix A).

- $\gamma_1 = \mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})$, $\gamma_2 = \mathbf{A}(\gamma_{2,1}\mathbf{U}\gamma_{2,2})$. Let M as in Fig. 16, with initial state s_0. It easy to see that $M \not\models \gamma$. Indeed, from s_0 start both a counterpath for $\mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})$ and a counterpath for $\mathbf{A}(\gamma_{2,1}\mathbf{U}\gamma_{2,2})$. The path $\pi_1 = [s_0, s_1, s_1, ...]$ is a counterpath for $\mathbf{A}(\gamma_{2,1}\mathbf{U}\gamma_{2,2})$, since the formula $\gamma_{2,2}$ is always false along it. Similarly, the path $\pi_2 = [s_0, s_2, s_2, ...]$ is a counterpath for $\mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})$, since the formula $\gamma_{1,2}$ is always false along it. On the other hand, π_1 cannot be a counterpath for $\mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})$, since therein $\gamma_{1,1}$ is always true and $\gamma_{1,2}$ is not always false. By symmetry, π_2 cannot be a counterpath for $\mathbf{A}(\gamma_{2,1}\mathbf{U}\gamma_{2,2})$. Hence, each counterexample for γ in M not linear. ∎

The main result of this paper on templates, Theorem 4.3, follows from Theorems 5.1 and 6.7.

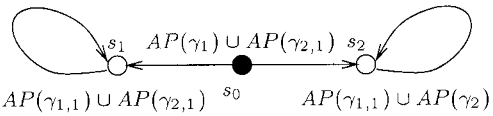

FIG. 16. Disjunction of $\gamma_1 = \mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1}\mathbf{U}\gamma_{2,2})$.

7. DISCUSSION AND CONCLUSION

For the class of ACTL formulas which are positive disjoint instantiations, the results in the preceding sections give a complete characterization of the c-linear fragment. This class is given by those formulas γ such that $\gamma^\star \in \mathbf{LIN}$. Observe that this class is efficiently recognizable.

This result can be extended by the same proof technique to more general classes of formulas γ, as long as certain independence properties hold on the pure state formulas. Introduce for each occurrence of a maximal pure state formula ϕ in γ a new propositional atom p_ϕ, and consider the formula

$$F(\gamma) = \bigwedge_{\phi \in MP(\gamma)} (p_\phi \leftrightarrow \phi),$$

where $MP(\gamma)$ is a list of all occurrences of maximal pure state formulas in γ. Call γ *pure state independent*, if for every truth value assignment to the atomic propositions p_ϕ, the formula $F(\gamma)$ is satisfiable. Observe that every positive disjoint instantiation γ is pure state independent.

Then, along the same line of proof as above we can show the following.

THEOREM 7.1. *Let γ be any pure state independent formula. Then, γ is c-linear if and only if $\gamma^\star \in \mathbf{LIN}$.*

However, testing pure state independence is complex in general; this amounts to evaluating the quantified Boolean formula (QBF) $\Phi = \forall P_\phi\, \exists AP.F(\gamma)$, where P_ϕ is the collection of all atomic propositions p_ϕ introduced for occurrences of maximal pure state formulas, and AP is the collection of all atomic propositions in γ. This problem is complete for the class Π_2^p of the polynomial hierarchy (cf. [8] for Π_2^p). Indeed, the evaluation of the QBFs $\forall X\, \exists Y.\psi$ is in Π_2^p [8], and the QBF Φ is constructible in polynomial time from γ. On the other hand, consider a QBF $\forall X\, \exists Y.\psi$, where ψ is of the form $y_1 \wedge \psi'$ where $y_1 \in Y$. Then, the ACTL formula $\gamma = (\mathbf{AX}x_1) \wedge \cdots \wedge (\mathbf{AX}x_n) \wedge (\mathbf{AX}\psi)$, where $X = \{x_1, ..., x_n\}$, is pure state independent, just if $\forall X\, \exists Y.\psi$ is true. Since deciding the latter is Π_2^p-hard, also deciding pure state independence is Π_2^p-hard.

Our results can be adapted for the concept of witness [5] in the existential fragment of *CTL* (denote this by *ECTL*), i.e., a portion of a computation tree which witnesses the truth of a formula $\mathbf{E}\phi$. Since on any structure M it holds that $M \models \mathbf{E}\phi$ if and only if $M \not\models \mathbf{A}\neg\phi$, the existence of linear witnesses (formally defined in the same vein as counterexamples) is related to the existence of linear counterexamples. As well-known [6], the equivalences $\neg\mathbf{A}(\phi\mathbf{V}\psi) = \mathbf{E}(\neg\phi\mathbf{U}\neg\psi)$ and $\neg\mathbf{A}(\phi\mathbf{U}\psi) = \mathbf{E}(\neg\phi\mathbf{V}\neg\psi)$ hold. It follows that a formula γ in the existential *CTL*-fragment has always a linear witness (call this *w-linear*), if and only if the formula obtained by dualization of γ and negating all elementary atomic propositions, is c-linear. As a consequence, all instantiations of an *ECTL*-template $\gamma^\star$ (defined as obvious) have linear witnesses (call this w-linear), just if the dual template $d(\gamma^\star)$ is c-linear. As a consequence, we obtain the following characterization of the class of w-linear *ECTL*-templates.

THEOREM 7.2. *Let $\gamma^\star$ be an ECTL-template. Then, $\gamma^\star$ is w-linear if and only if $d(\gamma^\star) \in \mathbf{LIN}$.*

In this paper, we have considered Kripke structures M in which the transition relation $R(M)$ is arbitrary. As already pointed out, many authors (e.g. [7, 10]) require that $R(M)$ is total. It appears that our main results (precisely, Theorems 4.1–4.3, 5.2, 5.3) and in particular Theorem 6.7 remain valid under restriction to the class of structures that have total transition relations. The structure M in the proof of Theorem 4.1 has total $R(M)$ by construction, and totality of $R(M)$ can be assumed to hold for M in the proof of Theorem 4.2 without loss of generality, since a graph having a node with no outgoing edges trivially has no Hamiltonian cycle. Furthermore, all structures in the proof of Theorem 6.7 that we have constructed for proving that certain formulas are not *c*-linear have total $R(M)$ if their constituents have.

Several issues remain for further work. One issue is the consideration of linear time operators which are derived from the basic operators **X**, **V**, **U**. The most important such operators are **F** (*sometimes*) and **G** (*globally*, *always*) defined as $\mathbf{F}\phi = \mathit{true}\mathbf{U}\phi$, $\mathbf{G}\phi = \mathit{false}\mathbf{V}\phi$. It is easily recognized from the definition of **LIN** and our results that these operators correspond to *c*-linear templates. However, the use of these templates in nesting, as well as the use of *true* and *false* in general, appears to be nontrivial. The characterization of the class of *c*-linear templates ACTL enriched by derived linear time operators and/or constants *true* and *false* is an interesting issue which remains to be explored.

Finally, an extension of our study by fairness constraints [4] would be interesting. In the general framework, path quantifiers do not range over all infinite paths, but instead over paths along which the fairness constraints, expressed by formulas, must be satisfied infinitely often. E.g., fair schedules in a system of concurrent infinite processes, represented through a Kripke structure, can be expressed easily through fairness constraints. Our results do not immediately carry over to this case. Techniques applied in [5, 9] might be useful.

APPENDIX: PROOFS

LEMMA 6.2. *For every positive disjoint instantiation γ of a template $\gamma^\star \in \mathbf{LIN}$, there exists a single-path structure M and a $k \geqslant 1$ such that $M \models \gamma$ and $\pi(M)^k$ is a local counterpath for γ (resp., $\pi(M)(k) \not\models \gamma$ if $\gamma^\star \in \mathscr{PSF}$).*

Proof. We prove the statement by induction on the stage $i \geqslant 0$ of $\Lambda^i = (S_1^i, S_2^i)$ in which $\gamma^\star$ first occurs.

(Basis) The case $i = 0$ is trivial.

(Induction) Assume that the statement holds for i and consider the possible cases for $\gamma^\star \in S_1^{i+1} \cup S_2^{i+1}$ where $i+1 > 0$. By the induction hypothesis, it remains to consider $\gamma^\star \notin S_1^i \cup S_2^i$.

- $\gamma^\star \in \mathscr{PSF}$. (In this case, $i = 1$.) Let M have the states s_0 and s_1, where s_0 is the unique initial state, and the transitions (s_0, s_1), (s_1, s_1). Let $L(M)(s_0) = AP(\gamma)$

and $L(M)(s_1) = \emptyset$. Clearly, M is a single-path structure such that $M \models \gamma$, and $M, \pi(M)^1(0) \not\models \gamma$. Thus the statement holds.

- $\gamma^\star \in \mathbf{AX}(S_1^i)$. Thus, $\gamma = \mathbf{AX}(\gamma_1)$. By the induction hypothesis, a single-path structure M with initial state s_0 and a $k \geqslant 1$ exist for γ_1 which satisfy the statement of the lemma. Let k^* be the least such k. If $k^* > 1$ we are done, since M is a single-path structure where also γ satisfies the statement of the lemma. Otherwise (i.e., if $k^* = 1$), we can modify M by adding a new state s_0' which reaches s_0 and has an arbitrary label. Denote by M' the resulting single-path structure with initial state s_0'. Since $\pi(M')^1 = \pi(M)$, it holds that $M' \models \gamma$. Furthermore, $\pi(M')^1$ is a local counterpath for γ, since $\pi(M')^2 = \pi(M)^1$. Hence the statement holds.

- $\gamma^\star \in \mathbf{AV}(\mathscr{PSF}, S_1^i)$. Let $\gamma = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$. By induction hypothesis, for γ_2 exist a single-path structure M and an index $k \geqslant 1$ such that the property of the lemma holds. We modify M by adding $AP(\gamma_1)$ to every state label in M. It is easy to see that the resulting structure M' satisfies $M' \models \gamma$ because γ_1 is globally true along $\pi(M')$. Furthermore, $\pi(M')^k$ is still a local counterpath for γ_2 (resp., $\pi(M')(k) \not\models \gamma_2$) since γ is a disjoint positive instantiation. Hence, the statement holds.

- $\gamma^\star \in \mathbf{AU}(S_1^i, \mathscr{PSF})$. Thus, $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$. Consider the single-path structure M with states s_0 and s_1, where s_0 is the initial state, transitions (s_0, s_1), (s_1, s_1) and labeling $L(M)(s_0) = AP(\gamma_2)$ and $L(M)(s_1) = \emptyset$. This M and $k = 1$ prove the statement for γ. Indeed, $M \models \gamma$ since γ_2 is true in s_0. Further, $\pi(M)^1$ is a local counterpath for γ since γ_2 is globally false along it.

- $\gamma^\star \in \mathbf{AU}(\mathscr{PSF}, S_2^i)$. Thus, $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$. By induction hypothesis, for γ_2 exist a single-path structure M and an index $k \geqslant 1$ as in the lemma. Without loss of generality, no atomic proposition from $AP(\gamma_1)$ occurs in any state label of M. Since γ is a positive disjoint instantiation, it is easy to see that M and k witness the statement also for γ. Indeed, $M \models \gamma$ since γ_2 is true in the initial state of M. Furthermore, $\pi(M)^k$ is a local counterpath for γ, since it is a local counterpath for γ_2 (resp., γ_2 is false in $\pi(M)(k)$) and γ_1 is globally false along it.

- $\gamma^\star \in S_1^i \vee \mathscr{PSF} \cup \mathscr{PSF} \vee S_1^i$. Thus, $\gamma = \gamma_1 \vee \gamma_2$. Assume that $\gamma_1^\star \notin \mathscr{PSF}$; the case $\gamma_2^\star \notin \mathscr{PSF}$ is similar. By induction hypothesis, for γ_1 exist a single-path structure M and an index $k \geqslant 1$ as stated in the lemma. Without loss of generality, no atomic proposition from $AP(\gamma_2)$ occurs in any state label of M. Since γ is a positive disjoint instantiation, it is easy to see that M and k witness the statement also for γ. Indeed, $M \models \gamma$ since $M \models \gamma_1$. Further, $\pi(M)^k$ is a local counterpath for γ since it is a local counterpath for γ_1 (resp., γ_1 is false in $\pi(M)(k)$) and γ_2 is globally false along it. Thus, the statement holds.

- $\gamma^\star \in S_1^i \wedge S_1^i$. Thus, $\gamma = \gamma_1 \wedge \gamma_2$, and w.l.o.g. $\gamma_1^\star \notin \mathscr{PSF}$. By induction hypothesis, for γ_1 exist a single-path structure M and an index $k \geqslant 1$ as stated in the lemma. We modify M by adding to every state label the set of atomic propositions appearing in γ_2. It is easy to see that the resulting structure M' and k witness the statement also for γ. Clearly, $M' \models \gamma$ since $M' \models \gamma_1$ and $M' \models \gamma_2$ since γ_2 is globally true in M'. Furthermore, $\pi(M')^k$ is a local counterpath for γ since it is a local counterpath for γ_1. Thus, the statement holds. This concludes the proof. ▮

LEMMA 6.3. *Each positive disjoint instantiation* γ *of any template* $\gamma^\star \in \mathbf{LIN}$ *has some single-path k-structure M.*

Proof. By induction on the stage $i \geqslant 0$ of $\Lambda^i = (S_1^i, S_2^i)$ in which $\gamma^\star$ first occurs.

(Basis) The case $i = 0$ is trivial.

(Induction) Assume that the statement holds for i, and consider the possible cases for $\gamma^\star \in S_1^{i+1} \cup S_2^{i+1}$, where $i+1>0$. By the induction hypothesis, it remains to consider $\gamma^\star \notin S_1^i \cup S_2^i$.

- $\gamma^\star \in \mathscr{PSF}$. (In this case, $i=1$.) Let M have the states s_0 and s_1, where s_0 is the unique initial state, and the transitions (s_0, s_1), (s_1, s_1). Let $L(M)(s_0) = \varnothing$ and $L(M)(s_1) = AP(\gamma)$. Clearly, M is a single path structure such that $M \not\models \gamma$, and $M, \pi(M)^1(0) \models \gamma$. Thus the statement holds.

- $\gamma^\star \in \mathbf{AX}(S_1^i)$. Let $\gamma = \mathbf{AX}(\gamma_1)$. By induction hypothesis, there exist a single-path structure M and an index $k \geqslant 1$ such that $M \not\models \gamma_1$ and $M, \pi(M)^i(0) \models \gamma_1$ for all $i \leqslant k$. Let s_0 be the initial state of M. We modify M by changing the initial state to a new state s with arbitrary label and adding the transition (s, s_0). Clearly, the resulting structure M' is single-path and $M' \not\models \gamma$. From the induction hypothesis, it follows that for each $i \geqslant k+1$, $M', \pi(M')^i(0) \models \gamma_1$. Hence, the statement holds.

- $\gamma^\star \in \mathbf{AV}(\mathscr{PSF}, S_1^i)$. Let $\gamma = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$. Let s_0 be the initial state of a single-path structure M for γ_2 and $k \geqslant 1$ as stated in the lemma, which exist by the induction hypothesis. Since $M \not\models \gamma_2$, it follows $M \not\models \gamma$. Furthermore, $M, \mu(\Pi)^i(0) \models \gamma_2$ implies $M, \mu(\Pi)^i(0) \models \gamma$, for each $i \geqslant k$. Thus the statement holds.

- $\gamma^\star \in \mathbf{AU}(S_1^i, \mathscr{PSF})$. Let $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$. Let for γ_1 be M and $k \geqslant 1$ as stated in the lemma, which exist by induction hypothesis. Without loss of generality, M includes $AP(\gamma_2)$ in each state label $L(s)$ except for the initial state s_0, which contains no atomic proposition from $AP(\gamma_2)$. Then, $M, s_0 \not\models \gamma_2$, and since $M \not\models \gamma_1$, it follows $M \not\models \gamma$. Furthermore, $M, \pi(M)^i(0) \models \gamma$ for all $i \geqslant k$ since γ_2 is true in $\pi(M)^i(0)$. Thus, the statement holds.

- $\gamma^\star \in \mathbf{AU}(\mathscr{PSF}, S_2^i)$. Let $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$. Let for γ_2 be M and $k \geqslant 1$ as stated in the lemma, whose existence follows from the induction hypothesis. Without loss of generality, we assume that the initial state s_0 of M contains no atomic proposition from $AP(\gamma_1)$. Since $M \models \gamma_2$, it follows $M \not\models \gamma$. Furthermore, since $M, \pi(M)^i(0) \models \gamma_2$ it follows that $M, \pi(M)^i(0) \models \gamma$, for all $i \geqslant k$. Thus the statement holds.

- $\gamma^\star \in S_1^i \vee \mathscr{PSF} \cup \mathscr{PSF} \vee S_1^i$. Let $\gamma = \gamma_1 \vee \gamma_2$. Assume $\gamma_1^\star \notin \mathscr{PSF}$; the case $\gamma_2^\star \notin \mathscr{PSF}$ is similar. Let for γ_1 be M and $k \geqslant 1$ as stated in lemma, which exist by induction hypothesis. Assume without loss of generality that no atomic proposition from $AP(\gamma_2)$ occurs in any label of M. Then, it is easy to see that M and k witness the statement for γ.

- $\gamma^\star \in S_1^i \wedge S_1^i$. Let $\gamma = \gamma_1 \wedge \gamma_2$. Let for γ_1 be M and $k \geqslant 1$ as stated in the lemma, which exist by the induction hypothesis. Assume without loss of generality that $\gamma_1^\star \notin \mathscr{PSF}$, and that each label of M includes $AP(\gamma_2)$. Since γ_2 is globally true in M, it is easy to see that M and k witness the statement also for γ. This concludes the proof. ∎

THEOREM 6.6. *Let γ be a positive disjoint instantiation of a template $\gamma^\star \in \mathbf{T}^\star$ such that either*

1. $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, *where* $\gamma_1^\star \notin \mathscr{PSF}$ *and* $\gamma_2^\star \in \mathbf{LIN} \setminus \mathscr{PSF}$, *or*
2. $\gamma = \mathbf{A}(\gamma_1 \mathbf{V} \gamma_2)$, *where* $\gamma_1^\star \notin \mathscr{PSF}$ *and* $\gamma_2^\star \in \mathbf{LIN}$.

Then, γ is not c-linear.

Proof. (1) The following six cases remain.

• $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$. We modify M in the following way. We add to every state s of M_1 except s_0 the set $AP(\gamma_2)$. Similarly, we add to every state of M_2 except s_0 the set $AP(\gamma_1)$. Finally, we add in every other state of $\Pi_{\gamma_{2,1}}$ in M_2 (see definition of right-structure), including s_0, the set $AP(\gamma_{2,2})$ (see Fig. 17). It easy to see that after these additions, $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$ still hold. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_1 is in M_2. Indeed, for every multi-path Π in M_2, $\Pi(1)$ cannot be a counterexample for $\gamma_{1,1}$, since each state of M_2 except contains the set $AP(\gamma_{1,1})$. Finally, no counterexample for γ_2 is in M_1. Indeed, a counterexample for γ_2 must contain a counterexample for $\gamma_{2,2}$. However, this is impossible, since $\gamma_{2,2}$ is globally true in M_1. Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. By Definition 3.5 a counterexample for γ must involve counterexamples for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. Clearly, this concludes the proof. Towards a contradiction, suppose Π is a linear counterexample involving only counterexamples for γ_2. By Definition 3.5, Π is such that $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$. But such a counterexample cannot be linear. Indeed, Π cannot lead into Π_{γ_2}, since it is a k-structure of γ_2. On the other hand, it cannot lead into M_1 or $\Pi_{\gamma_{2,1}}$, since a counterexample for γ_2 must contain a counterexample for $\gamma_{2,2}$, and $\gamma_{2,2}$ is globally true in $\Pi_{\gamma_{2,1}}$. Hence, every counterexample for γ in M is not linear.

• $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{U} \gamma_{2,2})$. We modify M as follows. We add to every state of M_1 except s_0 the set $AP(\gamma_2)$, and to every state of M_2 except s_0 the

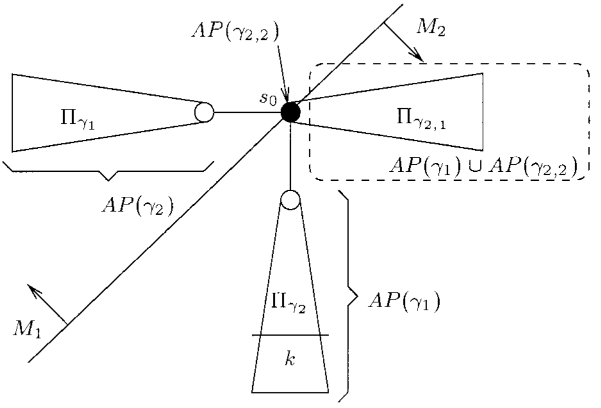

FIG. 17. The **X-V** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$.

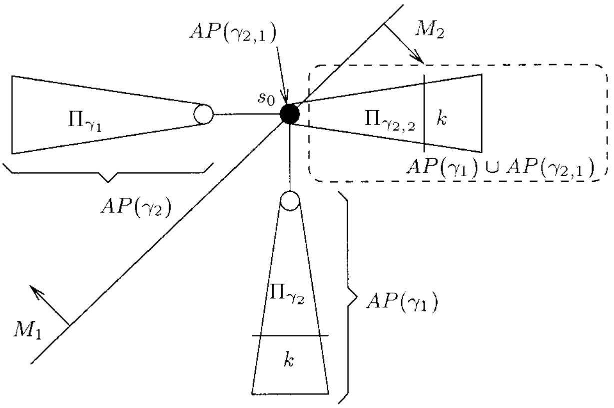

FIG. 18. The **X-U** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{U} \gamma_{2,2})$.

set $AP(\gamma_1)$. Finally, we add in every state of $\Pi_{\gamma_{2,2}}$ in M_2 including s_0 the set $AP(\gamma_{2,1})$ (see Fig. 18).

It easy to see that after these additions $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$ still hold. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_1 is in M_2. Indeed, for every multi-path Π in M_2, $\Pi(1)$ cannot be a counterexample for $\gamma_{1,1}$, since each state of M_2 except s_0 contains the set $AP(\gamma_{1,1})$. Finally, no counterexample for γ_2 is in M_1. Indeed, since each state of M_1 contains $AP(\gamma_{2,1})$, a counterexample for γ_2 in M_1 could only be a multi-path Π such that $\Pi(i)$ is a counterexample for $\gamma_{2,2}$, for each $i \geqslant 0$. But this is impossible, since for every multi-path Π in M_1, each state appearing in $\Pi(i)$, for $i \geqslant 1$ contains $AP(\gamma_{2,2})$. Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. Definition 3.5 request that a counterexample for γ must involve a counterexample for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. This, clearly, concludes the proof. Towards a contradiction, suppose Π is a linear counterexample for γ such that $\Pi(i)$ is a counterexample for γ_2, for every $i \geqslant 0$. But such a counterexample cannot be linear. Indeed, it can neither lead into M_1 nor into Π_{γ_2}, since this is a k-structure of γ_2. Furthermore, it cannot lead into $\Pi_{\gamma_{2,2}}$. Indeed, a counterexample for γ_2 cannot involve a counterexample for $\gamma_{2,1}$ as $\Pi_{\gamma_{2,2}}$ contains in each state the set $AP(\gamma_{2,1})$. Thus, such a counterexample could only be a multi-path Π such that $\Pi(i)$ is a (linear) counterexample for $\gamma_{2,2}$, for each $i \geqslant 0$. But this is not possible, since $\Pi_{\gamma_{2,2}}$ is a k-structure of $\gamma_{2,2}$. Hence, no counterexample for γ in M is linear.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{V} \gamma_{1,2})$ and $\gamma_2 = \mathbf{AX}(\gamma_{2,1})$. We modify M as follows. We add to every state of M_1 except s_0 the set $AP(\gamma_2)$ and to every state of M_2 except s_0 the set $AP(\gamma_1)$. Moreover, we add to s_0 the set $AP(\gamma_{1,2})$. Finally, we add in every other state of $\Pi_{\gamma_{1,1}}$ in M_1 (see definition of left-structure), the set $AP(\gamma_{1,2})$ (see Fig. 19).

After these additions, $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$ still hold. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_2 is in M_1. Indeed, for every multi-path Π in M_1, $\Pi(1)$ cannot be a counterexample for $\gamma_{2,1}$, since each state of M_1 except s_0 contains the set $AP(\gamma_{2,1})$. Finally, no counterexample for γ_1 is in M_2. Indeed, a counterexample for

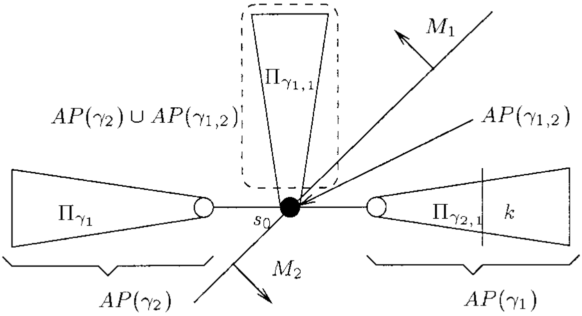

FIG. 19. The **V-X** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{V} \gamma_{1,2})$ and $\gamma_2 = \mathbf{AX}(\gamma_{2,1})$.

γ_1 must contain a counterexample for $\gamma_{1,2}$, but this is impossible, since each state in M_2 contains $AP(\gamma_{1,2})$. Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. By Definition 3.5, a counterexample for γ must involve counterexamples for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. This, clearly, concludes the proof. Towards a contradiction, suppose Π is a linear counterexample involving only counterexamples for γ_2. Definition 3.5 implies that $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$. But such a counterexample cannot be linear. Indeed, Π cannot lead into M_1 and not into M_2, since $\Pi_{\gamma_{2,1}}$ is a k-structure of $\gamma_{2,1}$. This proves the statement.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{V} \gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$. We modify M as follows. We add to every state of M_1 except s_0 the set $AP(\gamma_2)$. Then, we add to every state of M_2 except s_0 the set $AP(\gamma_1)$. Moreover, we add to s_0 the set $AP(\gamma_{1,2})$. Finally, we add in every state of $\Pi_{\gamma_{2,1}}$ in M_2, including s_0, the set $AP(\gamma_{2,2})$ (see Fig. 20).

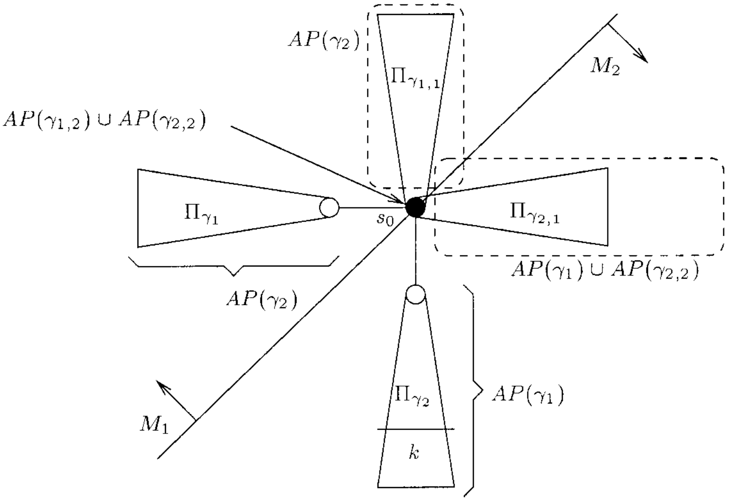

FIG. 20. The **V-V** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{V} \gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$.

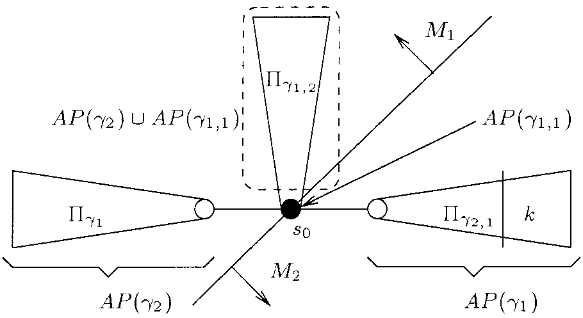

FIG. 21. The **U-X** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{U} \gamma_{1,2})$ and $\gamma_2 = \mathbf{AX}(\gamma_{2,1})$.

It easy to see that, also after these additions, $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_1 is in M_2. Indeed, $\gamma_{1,2}$ is globally true in M_2. Similarly, no counterexample for γ_2 is in M_1. Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear.

By Definition 3.5, a counterexample for γ must involve a counterexample for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. This, clearly, concludes the proof. Towards a contradiction, suppose Π is a linear counterexample for γ_2 such that $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$. But such a counterexample cannot be linear. Indeed, Π cannot lead into M_1, and it cannot lead into Π_{γ_2} since it is a k-structure of γ_2. On the other hand, it cannot lead into $\Pi_{\gamma_{2,1}}$, since a counterexample for γ_2 must contain a counterexample for $\gamma_{2,2}$ and $\gamma_{2,2}$ is globally true in $\Pi_{\gamma_{2,1}}$. Hence, every counterexample for γ in M is not linear.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{U} \gamma_{1,2})$ and $\gamma_2 = \mathbf{AX}(\gamma_{2,1})$. We modify M in the following way. We add to every state of M_1 except s_0 the set $AP(\gamma_2)$ and to every state of M_2 except s_0 the set $AP(\gamma_1)$. Finally, we add in every state of $\Pi_{\gamma_{1,2}}$ in M_1 the set $AP(\gamma_{1,1})$ (see Fig. 21).

It easy to see that, also after these additions, $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_2 is in M_1. Indeed, for every multi-path Π in M_1, $\Pi(1)$ cannot be a counterexample for $\gamma_{2,1}$, since each state of M_1 except s_0 contains the set $AP(\gamma_{2,1})$. Finally, no counterexample for γ_1 is in M_2. Indeed, since each state of M_2 contains the set $AP(\gamma_{1,1})$, a counterexample for γ_1 in M_2 could only be a multi-path Π such that each element $\Pi(i)$ is a counterexample for $\gamma_{1,2}$, for each $i \geqslant 0$. But this is impossible, since for every multi-path Π in M_2, each state appearing in $\Pi(i)$ contains the set $AP(\gamma_{1,2})$, for each $i \geqslant 1$. Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. By Definition 3.5 a counterexample for γ must involve a counterexample for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. This, clearly, concludes the proof. Towards a contradiction, suppose Π is a linear counterexample such that $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$. But such a counterexample cannot be linear. Indeed, Π cannot lead into M_1, since γ_2 is globally true in M_1, and it cannot lead into M_2, since $\Pi_{\gamma_{2,1}}$ is a k-structure of $\gamma_{2,1}$. Thus, the statement is proven.

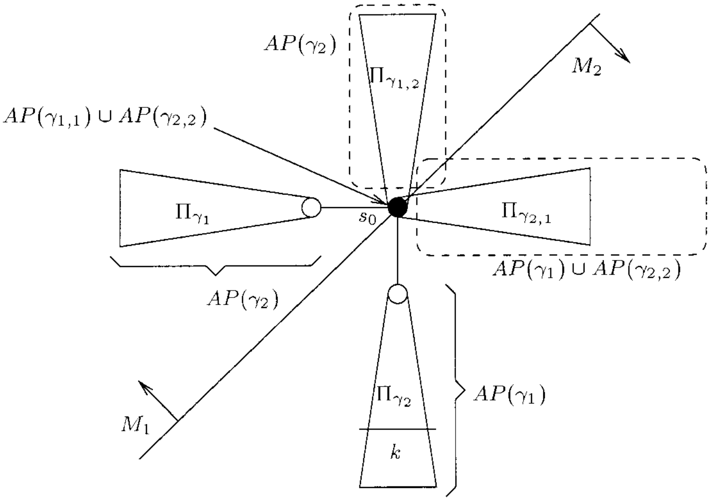

FIG. 22. The **U-V** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{U} \gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{U} \gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$. We modify M in the following way. We add to every state of M_1 except s_0 the set $AP(\gamma_2)$. Then, we add to every state of M_2 except s_0 the set $AP(\gamma_1)$. Moreover, we add to s_0 the set $AP(\gamma_{1,1}) \cup AP(\gamma_{2,2})$. Finally, we add to every other state of $\Pi_{\gamma_{2,1}}$ in M_2 the set $AP(\gamma_{2,2})$ (see Fig. 22).

It easy to see that after these additions, $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$ hold. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_1 is in M_2. Indeed, $\gamma_{1,1}$ is globally true in M_2 and for every multi-path Π in M_2, $\Pi(i)$, for $i \geqslant 1$, cannot be a counterexample for $\gamma_{1,2}$, since each state of M_2 except s_0 contains the set $AP(\gamma_{1,2})$. Finally, no counterexample for γ_2 is in M_1. Indeed, $\gamma_{2,2}$ is globally true in M_1. Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. By Definition 3.5, a counterexample for γ must involve a counterexample for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. This, clearly, concludes the proof. Towards a contradiction, suppose Π is a linear counterexample for γ such that $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$. Such a counterexample cannot be linear. Indeed, Π can neither lead into M_1 (cf. above) nor into $\Pi_{\gamma_{2,1}}$, since a counterexample for γ_2 must contain a counterexample for $\gamma_{2,2}$ which is globally true in $\Pi_{\gamma_{2,1}}$. Furthermore, Π cannot lead into Π_{γ_2}, since it is a k-structure of γ_2. Hence, no counterexample for γ in M is linear.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{U} \gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{U} \gamma_{2,2})$. We modify M in the following way. We add to every state of M_1 except s_0 the set $AP(\gamma_2)$ and to every state of M_2 except s_0 the set $AP(\gamma_1)$. Moreover, we add in s_0 the set $AP(\gamma_{1,1}) \cup AP(\gamma_{2,1})$. Finally, we add in every other state of $\Pi_{\gamma_{2,2}}$ in M_2 (see definition of right-structure) the set $AP(\gamma_{2,1})$ (see Fig. 23).

It easy to see that after these additions, $M_1 \not\models \gamma_1$ and $M_2 \not\models \gamma_2$ hold. Thus, $M \not\models \gamma$. Moreover, no counterexample for γ_1 is in M_2. Indeed, $\gamma_{1,1}$ is globally true in M_2 and for every multi-path Π in M_2, $\Pi(i)$, for $i \geqslant 1$, cannot be a counterexample for

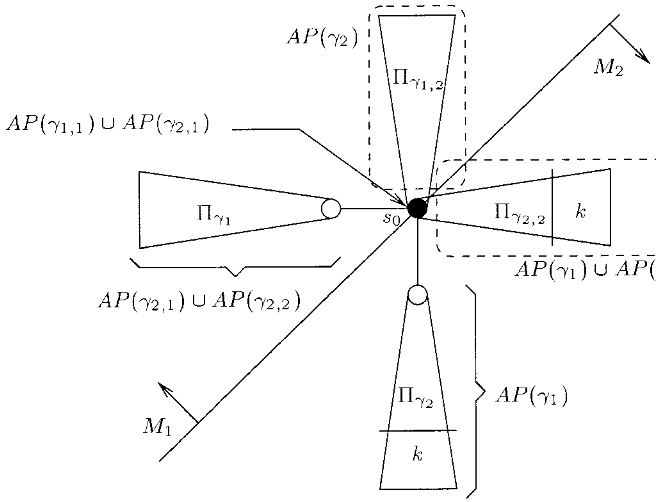

FIG. 23. The **U**-**U** case: $\mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{U} \gamma_{1,2})$ and $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{U} \gamma_{2,2})$.

$\gamma_{1,2}$, since each state of M_2 except s_0 contains the set $AP(\gamma_{1,2})$. Similarly, no counterexample for γ_2 is in M_1. Hence, a counterexample for γ involving counterexamples for both γ_1 and γ_2 cannot be linear. By Definition 3.5, a counterexample for γ must involve a counterexample for γ_2. Now we show that every counterexample for γ involving only counterexamples for γ_2 is not linear. This, clearly, concludes the proof. Towards a contradiction, let Π be a linear counterexample for γ such that $\Pi(i)$ is a counterexample for γ_2, for each $i \geqslant 0$. But such a counterexample cannot be linear. Indeed, Π cannot lead into M_1, and furthermore, it cannot lead into Π_{γ_2}, since this is a k-structure of γ_2. Finally, it also cannot lead into $\Pi_{\gamma_{2,2}}$. Indeed, a counterexample for γ_2 cannot involve a counterexample for $\gamma_{2,1}$, as $\Pi_{\gamma_{2,2}}$ contains in each state the set $AP(\gamma_{2,1})$. Thus, such a counterexample could only be a multi-path Π such that $\Pi(i)$ is a (linear) counterexample for $\gamma_{2,2}$, for each $i \geqslant 0$. But this is not possible, since $\Pi_{\gamma_{2,2}}$ is a k-structure of $\gamma_{2,2}$. Hence, every counterexample for γ in M is not linear.

(2) The following two cases remain.

- $\gamma_1^\star = \mathbf{A}(\gamma_{1,1}^\star \mathbf{V} \gamma_{1,2}^\star)$. Then, $\gamma = \mathbf{A}(\mathbf{A}(\gamma_{1,1} \mathbf{V} \gamma_{1,2}) \mathbf{V} \gamma_2)$. To construct M, we modify the above structure M_γ as follows. We add in each state not appearing in $\pi(M')^k$ the set $AP(\gamma_2)$. Note that this addition does not affect the existence of counterexamples for γ_1 starting with $s_0, s_1, ..., s_{k-1}$, since $AP(\gamma_1)$ and $AP(\gamma_2)$ are disjoint. Finally, we add the set $AP(\gamma_{1,2})$ in every state appearing in $\pi(M')$. This addition preserves the existence of counterexamples for $\gamma_{1,1}$ (hence, for γ_1) starting with $s_0, s_1, ..., s_{k-1}$. Furthermore, $\pi(M')^k$ is still a local counterpath for γ_2, since $AP(\gamma_2)$ and $AP(\gamma_{1,2})$ are disjoint. The resulting conic structure with initial state s_0 is M (see Fig. 24).

It holds that $M \not\models \gamma$. Indeed, there exists a multi-path Π_2, such that $\Pi_2(i)$ is a ℓ-counterexample for γ_1, for $0 \leqslant i \leqslant k-1$ (recall that each state s_i is origin of a

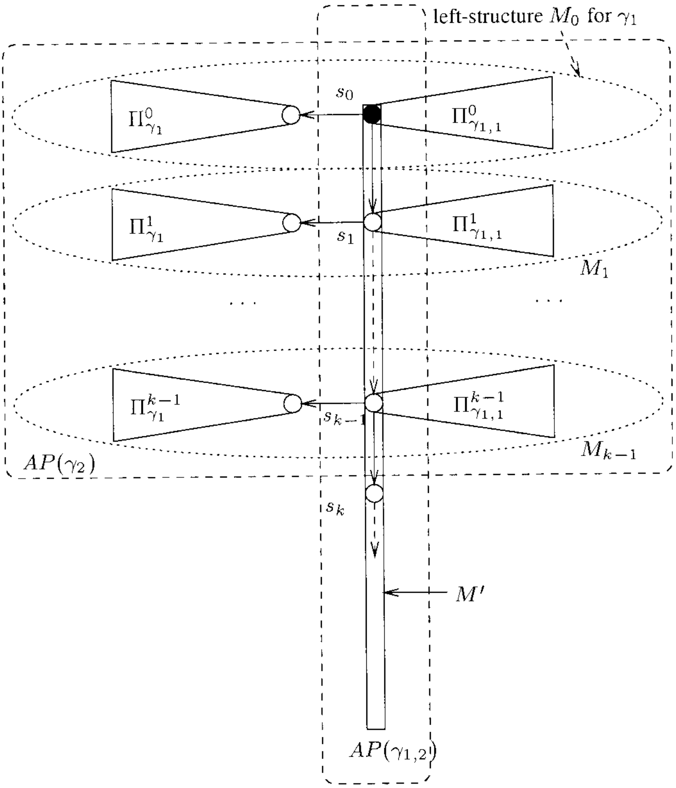

FIG. 24. Nesting into unless, the **V** case: $\gamma = \mathbf{A}(\mathbf{A}(\gamma_{1,1}\mathbf{V}\gamma_{1,2})\mathbf{V}\gamma_2)$.

ℓ-counterexample for γ_1), and $\Pi_2(k)$ is a local counterexample for γ_2 with main path $\pi(M')^k$. Clearly, this multi-path is not linear. Moreover, no linear counterexample for γ in M exists. Indeed, each counterexample for γ needs a counterexample for γ_2. It holds that every path starting at s_0 cannot be a counterpath for γ_2. Indeed, each path π not reaching the state s_k cannot be a counterpath for γ_2, since the label of each state appearing in π would contain the set $AP(\gamma_2)$.

On the other hand, the only path starting at s_0 and reaching s_k is $\pi(M')$. As M' was chosen according to Lemma 6.2, this path cannot be a counterpath for γ_2 by construction. Hence, we need a counterexample such that the first element is a counterexample for γ_1. Clearly, we cannot find a counterexample for γ_1 along the path $\pi(M')$, since each state in it contains $AP(\gamma_{1,2})$ (and a counterexample for γ_1 necessarily contains a counterexample for $\gamma_{1,2}$). Hence, each counterexample for γ necessarily contains branching, that is, it is not linear.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})$, i.e., $\gamma = \mathbf{A}(\mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})\mathbf{V}\gamma_2)$. We modify the structure M_γ from above as follows. We add to each state not appearing in $\pi(M')^k$ the set $AP(\gamma_2)$. Note that this addition does not affect the existence of local counterexamples for γ_1 starting at $s_0, s_1, \ldots, s_{k-1}$, since $AP(\gamma_1)$ and $AP(\gamma_2)$ are disjoint. Furthermore, add the set $AP(\gamma_{1,1})$ in every state appearing in $\pi(M')$. This addition preserves the existence of counterexamples for $\gamma_{1,2}$ (hence for γ_1) starting at

$s_0, s_1, ..., s_{k-1}$. Finally, we add in every state appearing in $\pi(M)^k$ the set $AP(\gamma_{1,2})$. Clearly, after this addition $\pi(M')^k$ is still a local counterpath for γ_2, since $AP(\gamma_1)$ and $AP(\gamma_2)$ are disjoint. The resulting conic structure with initial state s_0 is M (see Fig. 25).

We can see that $M \not\models \gamma$. Indeed, there exists a multi-path Π_2, such that $\Pi_2(i)$ is a ℓ-counterexample for γ_1, for $0 \leqslant i \leqslant k-1$, and $\Pi_2(k)$ is a counterexample for γ_2 with main path $\pi(M')^k$. Clearly, this multi-path is not linear. Moreover, no linear counterexample for γ exists in M. Indeed, each counterexample for γ needs a counterexample for γ_2. Every path π starting at the initial state s_0 cannot be a counterpath for γ_2. Indeed, if π does not reach the state s_k, it cannot be a counterpath for γ_2, since the label of each state appearing in π would contain the set $AP(\gamma_2)$. On the other hand, the only path starting at s_0 and reaching s_k is $\pi(M')^k$. Since M' was chosen according to Lemma 6.2, it is not a counterpath for γ_2. Hence, we need a counterexample whose first element is a counterexample for γ_1. Clearly, we cannot find a counterexample for $\gamma_{1,1}$ along the path $\pi(M')$, since each state in it contains the set $AP(\gamma_{1,1})$. Hence, a counterexample for γ_1 could only be a multi-path Π such that $\Pi(i)$ is a counterexample for $\gamma_{1,2}$, for each $i \geqslant 0$. But such a counterexample cannot be found along the path $\pi(M')$. Indeed, along its suffix $\pi(M')^k$ the formula $\gamma_{1,2}$ is always true.

Hence, each counterexample for γ necessarily contains branching, that is, it is not linear. ▮

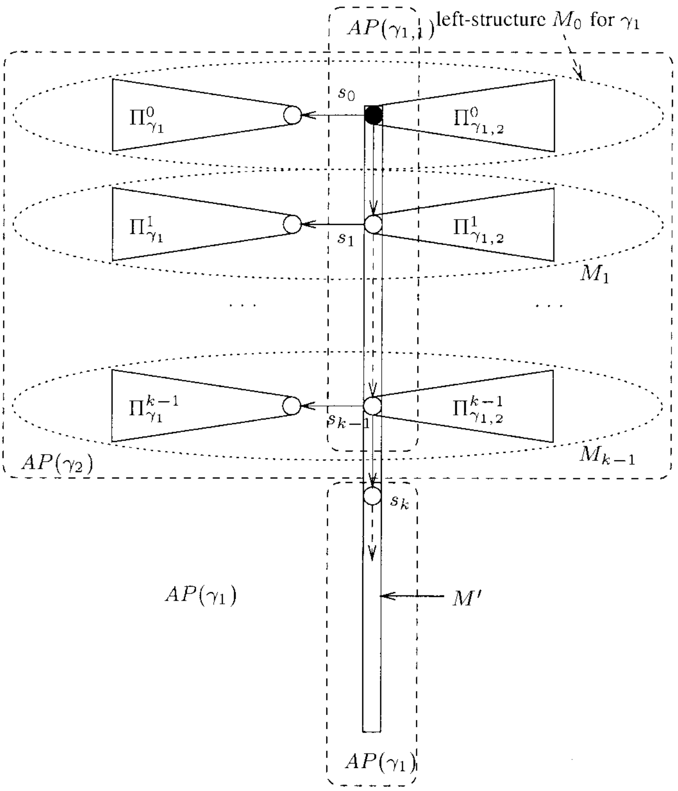

FIG. 25. Nesting into unless, the **U** case: $\gamma = \mathbf{A}(\mathbf{A}(\gamma_{1,1}\mathbf{U}\gamma_{1,2})\mathbf{V}\gamma_2)$.

Theorem 6.7. *Let γ be any positive disjoint instantiation of a template $\gamma^\star \in \mathbf{T}^\star$. If $\gamma^\star \notin \mathbf{LIN}$, then γ is not c-linear.*

Proof (continued).

- $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, where $n_{\mathbf{A}}(\gamma_1) + n_{\mathbf{A}}(\gamma_2) = k-1$.

2. $\gamma_1^\star \notin \mathbf{LIN}$ and $\gamma_2^\star \in \preccurlyeq\mathscr{PSF}$. Since $\gamma_1^\star \notin \mathbf{LIN}$, by the induction hypothesis a structure M exists such that $M \not\models \gamma_1$ and no counterexample for γ_1 in M is linear. Without loss of generality, M is conic with initial state s_0 and $AP(\gamma_2) \cap AP(M) = \emptyset$.

Clearly, $M \not\models \gamma$, since $M \not\models \gamma_2$. Modify now M by adding to each state except s_0 the set $AP(\gamma_2)$. Since $AP(\gamma_1) \cap AP(\gamma_2) = \emptyset$, still $M \not\models \gamma_1$ holds. Moreover, since $L(M)(s_0) \cap AP(\gamma_2) = \emptyset$, also $M \not\models \gamma_2$ holds. Thus, $M \not\models \gamma$. It holds that each counterexample for γ in M must contain a counterexample for γ_1, and thus it is not linear. Indeed, in any alternative counterexample Π for γ the element $\Pi(i)$ would a local counterexample for γ_2, for every $i \geqslant 0$. Since all states of M except s_0 contain $AP(\gamma_2)$, this is impossible.

3. $\gamma_2^\star \notin \mathbf{LIN}$. By the inductive hypothesis, there exists a structure M such that both $M \not\models \gamma_2$ and each counterexample for γ_2 in M is not linear. W.l.o.g., M is conic with initial state s_0 and $AP(\gamma_1) \cap AP(M) = \emptyset$.

Clearly, $M \not\models \gamma$, where $\gamma = \mathbf{A}(\gamma_1 \mathbf{U} \gamma_2)$, since $M \not\models \gamma_1$ and $M \not\models \gamma_2$. We can conclude that each counterexample for γ in M is not linear. Indeed, if Π is a counterexample for γ in M, $\Pi(0)$ must be a ℓ-counterexample for γ_2. Moreover $or(\Pi(0)) = s_0$. Hence, $\Pi(0)$ is a counterexample for γ_2 in M. Consequently, $\Pi(0)$ and hence also Π cannot be linear.

4.1. First observe that $M \not\models \gamma$. Indeed, each state s_i, for $0 \leqslant i \leqslant k$ is origin of a local counterexample for γ_2. Furthermore, s_k is also origin of a local counterexample for γ_1. It remains to show that no linear counterexample is in M. In any counterexample Π for γ the element $\Pi(0)$ must be a counterexample for γ_2. This implies that a counterpath for γ cannot reach state s_k. Indeed, the only path reaching state s_k is $\pi(M')$, which by construction is not a counterpath for γ_2. Thus, a counterpath π for γ could only lead into some structure M_i, where $0 \leqslant i \leqslant k-1$. However, in each M_i formula γ_1 is globally true. Hence π would have to satisfy that π^j, for each $j \geqslant 1$, is a local counterpath for γ_2. Since each M_i is a k-structure for γ_2, this is impossible. This proves that no linear counterexample for γ exists in M.

4.2. The path $[s_0, s_0', \dots]$ is a counterpath for γ_2. Thus, the multi-path $[[s_0, s_0', \dots], [s_0, s_0' \dots], \dots]$ is a counterexample for the γ. It holds that no linear counterexample for γ exists in M. Indeed, since $AP(\gamma_1)$ is contained in each state, any counterexample for γ must contain infinitely many counterexamples for γ_2. Since s_0' is the initial state of a k-structure for γ_2, no counterpath for γ is possible which reaches s_0'. Hence, the only possibility for a counterpath of γ is $\pi = [s_0, s_0, s_0, \dots]$. Since s_0 contains $AP(\gamma_{2,2})$, this is impossible. Thus, non linear counterexample for γ exists.

4.3. First observe that $M \not\models \gamma$. Indeed, each state s_i, for $0 \leqslant i \leqslant k-1$ is origin of a local counterexample for ϕ_1 and thus for γ_2. Furthermore, s_k is also origin of a local counterexample for ϕ_2, and then for γ_2. Moreover, s_k is a local counterexample for the formula γ_1.

Now we show that no linear counterexample for γ exists in M. By Definition 3.5, in any counterexample Π for γ the element $\Pi(0)$ must be a counterexample for γ_2. Hence, a counterpath for γ cannot reach state s_k. Indeed, the only path reaching state s_k is $\pi(M')$. This path is not a counterpath for γ_2 by construction: $\Pi(M')$ does not contain any local counterpath for ϕ_1, and, moreover, $\pi(M')$ is not an counterpath for ϕ_2. Thus, a counterpath π for γ could only lead into some structure M_i, where $0 \leqslant i \leqslant k-1$. Since in each M_i formula γ_1 is globally true, the suffix π^i must be a local counterpath for γ_2, for each $i \geqslant 1$. Since each state in M_i except the initial state s_i contains $AP(\phi_2)$, this counterpath for γ_2 can only be a counterpath for ϕ_1. But this is impossible, since a right-structure for formula ϕ_1 cannot contain a linear counterexample Π such that $\Pi(i)$ is a counterexample for ϕ_1, for each $i \geqslant 0$. Thus, it follows that no linear counterexample for γ exists in M.

4.4. It holds that $M \not\models \gamma$. Indeed, there exists a counterexample Π for γ where $\Pi(0)$ is a counterexample for γ_2, and $\Pi(1)$ is a counterexample for both γ_1 and γ_2. Furthermore, no linear counterexample for γ exists in M. To see this, observe that no path π leading into Π_{ϕ_2} or into $\Pi_{\phi_{2,2}}$ can be a counterpath for γ, as γ_1 and ϕ_1 are always true there and Π_{ϕ_2}, $\Pi_{\phi_{2,2}}$ are k-structures for $\phi_{2,2}$ (consequently, ϕ_2 is not globally false). Thus, only $\pi = [s_0, s_1, s_2, s_2, \ldots]$ remains as a candidate for a counterpath for γ. To eliminate π, assume towards a contradiction that $\pi = \mu(\Pi)$ for some linear counterexample Π for γ. The first element $\Pi(0)$ of every counterexample Π for γ must be a counterexample for $\gamma_2 = \phi_1 \wedge \phi_2$; since ϕ_1 is true in s_0, it must be a counterexample of ϕ_2. Along π, however, $\phi_{2,2}$ is not always false, which means that $\Pi(0)$ must involve a counterexample for $\phi_{2,1}$. Along π, however, $\phi_{2,1}$ is by construction always true. This raises a contradiction, and proves that in M no linear counterexample for γ exists.

- $\gamma^\star = \phi_1^\star \wedge \phi_2^\star$ or $\gamma^\star = \phi_1^\star \vee \phi_2^\star$, where $n_{\mathbf{A}}(\phi_1^\star) + n_{\mathbf{A}}(\phi_2^\star) = k$. In the remaining five cases, the labeling of M is chosen as follows (the suitability of M is easily verified):

- $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{U} \gamma_{1,2})$, $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$. Set $L(M)(s_0) = AP(\gamma_{1,1}) \cup AP(\gamma_{2,2})$, $L(M)(s_1) = AP(\gamma_1)$, and $L(M)(s_2) = AP(\gamma_{1,1}) \cup AP(\gamma_{2,2})$.

- $\gamma_1 = \mathbf{A}(\gamma_{1,1} \mathbf{V} \gamma_{1,2})$, $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$. Set $L(M)(s_0) = AP(\gamma_{1,2}) \cup AP(\gamma_{2,2})$, $L(M)(s_1) = AP(\gamma_{1,2})$, and $L(M)(s_2) = AP(\gamma_{2,2})$.

- $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$, $\gamma_2 = \mathbf{AX}(\gamma_{2,1})$. Set $L(M)(s_0) = \varnothing$, $L(M)(s_1) = AP(\gamma_{1,1})$, and $L(M)(s_2) = AP(\gamma_{2,1})$.

- $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$, $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{U} \gamma_{2,2})$. Set $L(M)(s_0) = AP(\gamma_{2,1})$, $L(M)(s_1) = AP(\gamma_1)$, and $L(M)(s_2) = AP(\gamma_2)$.

- $\gamma_1 = \mathbf{AX}(\gamma_{1,1})$, $\gamma_2 = \mathbf{A}(\gamma_{2,1} \mathbf{V} \gamma_{2,2})$. Set $L(M)(s_0) = AP(\gamma_{2,2})$, $L(M)(s_1) = AP(\gamma_{1,1})$, and $L(M)(s_2) = AP(\gamma_{2,2})$.

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